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SEMI-ANALYTICAL FINITE ELEMENT METHOD FOR SOLVING THREE-DIMENSIONAL PROBLEMS OF THERMO-VISCO-ELASTO-PLASTICITY OF PRISMATIC BODIES WITH MATERIAL DAMAGE CONSIDERATION

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A methodology for solving three-dimensional problems of thermo-visco-elasto-plasticity of prismatic bodies, taking into account material damage, has been developed based on the semi-analytical finite element method (SAFEM). The methodology includes governing relations of the SAFEM for an oblique prismatic inhomogeneous finite element, accounting for variations in the components of the metric tensor in the plane of its cross-section, as well as a step-by-step solution algorithm based on displacement extrapolation. The efficiency of the methodology and the reliability of the obtained results are demonstrated using benchmark test examples.

Keywords: semi-analytical finite element method, methodology, prismatic inhomogeneous finite element, metric tensor.

Introduction. Many critical elements of mechanical engineering structures have the form of spatial prismatic bodies with complex geometry, subjected to force and thermal loads arbitrarily distributed in space and time. Evaluating their load-bearing capacity requires solving three-dimensional problems of thermo-visco-elasto-plasticity, which in practice is mainly possible through the use of numerical methods. Among these, the finite element method (FEM) and its efficient modifications, in particular the semi-analytical finite element method (SAFEM), occupy an important place. To date, significant experience has been accumulated in the use of SAFEM for solving a wide range of problems in the mechanics of deformable solids [1, 2, 3].

Taking into account nonlinear deformation processes, in particular plasticity and creep, requires the use of step-by-step loading simulation algorithms and iterative solution of the corresponding systems of equations at each step. The number of such steps during the simulation of long-term deformation under creep conditions may reach thousands. Considering the complex shape of the objects under study and the essentially three-dimensional nature of their stress–strain state, the number of unknowns in the problem may exceed hundreds of thousands. If the load changes over time, particularly due to the presence of a temperature field, the complexity of the problem and the computational cost increase approximately by an order of magnitude.

A significant reduction in the number of unknowns in the discrete model can be achieved by using skew finite elements (FE), whose governing relations take into account the variability of the components of the metric tensor in the plane of the element cross-section [4]. This necessitated the expansion of the finite element basis through the development of new FE modifications capable of accounting for such geometric features and enabling the modeling of arbitrary boundary conditions [4].

Another important direction for reducing computational costs is the use of efficient algorithms for solving systems of nonlinear equations. When analyzing three-dimensional problems of thermo-visco-elasto-plasticity, it is necessary to determine stresses taking into account not only plastic strains, creep, and material damage, but also thermal strains, and on this basis to form the vector of nodal reactions [4].

A nonhomogeneous oblique prismatic finite element with arbitrary boundary conditions. For the approximation of spatially non-homogeneous prismatic bodies, spatial non-homogeneous prismatic finite elements are used (Fig. 1), which represent a prism formed by translating a quadrilateral of arbitrary shape along a straight generating line.

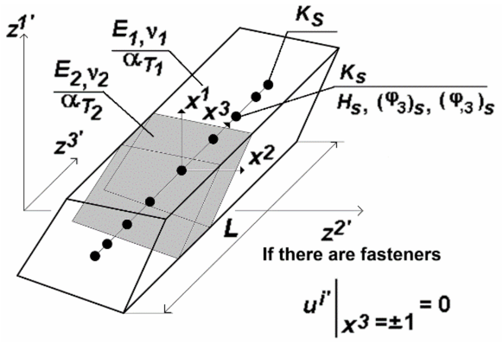


Fig. 1. Non-homogeneous prismatic finite element in the basic coordinate system

It is assumed that the components of the elastic constants tensor and the determinant of the matrix composed of the components of the metric tensor g_{ij}, as well as the coefficient of linear expansion α_T are equal to the values of the corresponding quantities at the center of the FE cross-section.

The distribution of displacements and temperature within the FE cross-section is described by a bilinear law.

Along the generating direction, the displacements, temperature, and their derivatives with respect to the generating coordinate x^3 are approximated by expansions in a system of coordinate functions using Lagrange and Michlin polynomials.

The adopted system of functions satisfies the conditions of completeness and linear independence and allows various types of boundary conditions at the ends of the body to be formulated in the simplest and most efficient way using the traditional FEM approach, i.e., by eliminating the corresponding equations [1].

The considered finite elements are oriented toward the analysis of a wide class of prismatic bodies. They must ensure not only high accuracy in representing the stress–strain state of structures of complex shape, but also a high rate of convergence of the results to the exact solution. As shown in [1], the application of the moment scheme of finite elements (MSFE) [5, 6] significantly increases the efficiency of numerical studies of spatial structures based on FEM. In addition, MSFE ensures the absence of strains during rigid-body displacement and eliminates the phenomenon of “parasitic shear” that arises in the analysis of thin-walled structures using spatial finite elements.

The expressions for the stiffness matrix and the vector of nodal reactions of the non-homogeneous skew prismatic finite element obtained in [4] make it possible to construct discrete models for non-homogeneous prismatic bodies of complex shape. Taking into account the variability of the metric tensor components in the cross-section of the prismatic FE reduces restrictions on the ratio of side dimensions and the skewness of the FE cross-section, which in turn makes it possible to decrease the number of unknowns in SAFEM discrete models.

Algorithm for Solving Three-Dimensional Problems of Thermo-Visco-Elasto-Plasticity. The solution of three-dimensional thermo-visco-elasto-plastic problems requires the use of efficient algorithms oriented toward modeling long-term material deformation processes. Such algorithms should be invariant with respect to the forms of the creep equations used and allow for the consideration of variations in the physical and mechanical properties of the material depending on the parameters of the stress–strain state and temperature.

The process of nonlinear deformation can be represented as a set of discrete steps with respect to the parameters of external loading and time. Thus, solving the problem requires the use of a step-by-step algorithm. In turn, at each step, iterative algorithms are applied to solve the systems of nonlinear equations arising from the finite element method.

The choice of the step size with respect to the parameters (time step Δt and load step Δp) is determined by the need to satisfy the convergence conditions both of the iterative process for solving the nonlinear systems of equations and of the obtained results. The values of Δt and Δp significantly depend on the mechanical characteristics of the material (parameters of elastic–plastic deformation and creep curves) as well as on the nature of the variation of the external load. The correct selection of the time and load steps substantially affects the accuracy of the obtained results; therefore, performing appropriate convergence studies is an important stage in solving the problem.

To account for variations in the physical and mechanical properties of the material depending on temperature and external loading, the possibility of updating these properties is provided at the beginning of each solution step. Within a single step, the physical and mechanical characteristics are assumed to remain constant.

The algorithm proposed in this work involves the implementation of two iterative cycles: an inner cycle associated with solving a system of linear equations at each iteration of the nonlinear cycle, and an outer cycle related to solving the nonlinear system of equations itself. The total number of iterations is equal to the product of the number of iterations in these two cycles, which significantly exceeds the number of iterations required for solving a nonlinear problem using the standard FEM.

Paper [4] presents a description of an algorithm for determining the parameters of the stress–strain state during thermo-visco-elasto-plastic deformation with consideration of material damage based on displacement extrapolation in prismatic bodies within the framework of the semi-analytical finite element method (SAFEM).

The nonlinear deformation process of spatial prismatic bodies is simulated using a step-by-step method with respect to the external load parameter and time. At each step, the Newton–Kantorovich iterative procedure is used to solve the system of nonlinear equations.

At the n iteration of the m step, the vector of unknown increments of displacement amplitudes $\{\Delta U_l\}_n^m$ of the SAFEM nonlinear system can be written as

$$\{\Delta U_l\}_n^m = \{\Delta U_l\}_{n-1}^m + \beta\,[K_{ll}]^{-1}\left(\{Q_l\}^m - \{R_l\}_n^m\right), \tag{1}$$

where $\{\Delta U_l\}_{n-1}^m$, $\{\Delta U_l\}_n^m$ - the coefficients represent the expansion of the vector of increments of nodal displacement amplitudes at iterations n-1 and n, respectively, within step m; β is the relaxation parameter ($1<\beta<2$), $\{Q\}^m$ is the vector of nodal loads; and $\{R\}_n^m$ is the vector of nodal reactions.

The parameters of the stress–strain state are determined sequentially. The values of thermal strains are calculated according to the moment finite element scheme, while plastic and creep strains are determined according to the prescribed material deformation laws.

When implementing algorithm (1), it was previously assumed that at the beginning of the first iteration of step m the increment of displacement amplitudes is equal to zero ($\{\Delta U_l\}_1^m = 0$). However, in the step-by-step solution procedure, considering the need to satisfy the convergence conditions of algorithm (1), the increments of the stress–strain parameters compared with their total values at two consecutive load steps (for plasticity problems) or time steps (for creep problems) are small.

Therefore, to increase the efficiency of the proposed algorithm and, accordingly, reduce computational costs, an approach based on the extrapolation of displacement increments was implemented. This approach uses the values obtained at the previous step $\{\Delta U_l\}^{m-1}$ and the ratio of the load increment parameters of the current step $\{\Delta Q_l\}^m$ and the previous step $\{\Delta Q_l\}^{m-1}$ for plasticity problems, or the ratio of time steps Δt_m and Δt_{m-1} for creep problems:

$$\{\Delta\hat{U}_l\}_1^m = \{\Delta U_l\}^{m-1}\frac{\{\Delta Q_l\}^m}{\{\Delta Q_l\}^{m-1}},\quad \{\Delta\hat{U}_l\}_1^m = \{\Delta U_l\}^{m-1}\frac{\Delta t_m}{\Delta t_{m-1}}. \tag{2}$$

Using the obtained displacement increments, the nodal reactions $\{R_l\}_1^m$ are calculated, which are subsequently used in the iterative process.

Efficiency and Reliability of Solving Thermo-Visco-Elasto-Plastic Problems

Thermo-Elasto-Plastic Deformation of a Cube. To study the convergence behavior and reliability of the results obtained using the proposed algorithm in the presence of plastic deformation, we consider the problem of deformation of a non-uniformly heated cube. A thermo-elasto-plastic stress state of a cube with dimensions 10×10×10 mm is analyzed. The cube is initially in a natural state at temperature $T_0 = 20^\circ C$, after which it is subjected to non-uniform heating in the absence of body and surface forces [7].

Since the cube has three planes of symmetry, only one eighth of the cube was used for the analysis. The corresponding discrete model is shown in Fig. 2.

The temperature field within the cube volume is described by the function:

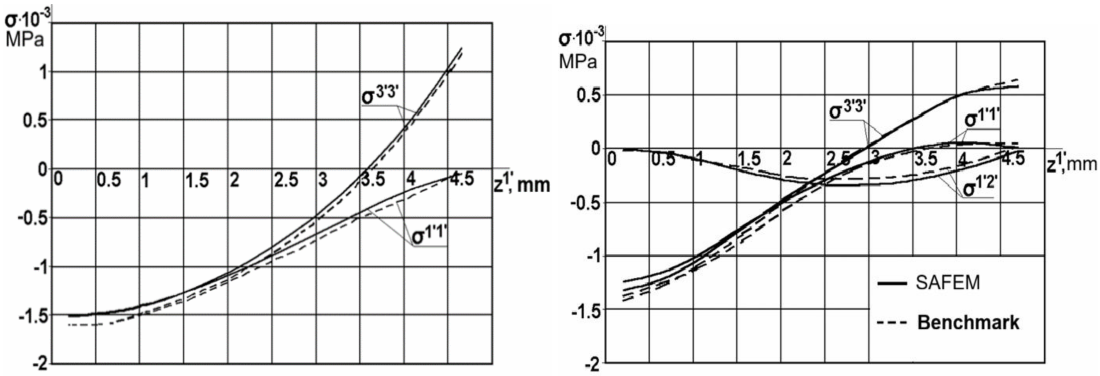
$$T = 800\cos\pi z^{1'}\cos\pi z^{2'}\cos\pi(z^{3'}+2.5).$$

The mechanical properties are assumed to be independent of temperature and have the following values: $E = 1.96\cdot 10^5\,\text{MPa}$, $\nu = 1/3$, $\alpha_T = 1.5\cdot 10^{-5}$ degrees^{-1}.

Figure 3 shows the variation of stress components as a function of coordinate $z^{1'}$:

for elastic deformation of the cube (Fig. 3 (a)) at $z^{2'} = 0.625$ mm, $z^{3'} = -1.875$ mm;

for elasto-plastic deformation (Fig. 3 (b)) at $z^{3'} = -1.875$ mm, $z^{1'} = z^{2'}$.



(a) (b)

Fig. 3. Variation of stresses in the cross-section of the cube

Comparison of the results for determining the stress parameters of the cube under elastic and elasto-plastic deformation obtained using the semi-analytical finite element method (SAFEM) with the results presented in [7] shows that in the region of maximum stress values the discrepancy does not exceed 5%.

Analysis of the obtained results shows that the normal stresses reach their maximum values at the center of the cube and near its surface. In the center of the cube, a stress state close to hydrostatic compression occurs, while the reduction in stress level due to the development of plastic strains reaches 35%.

A comparison of the number of iterations required to solve the problem at different deformation stages using algorithms with displacement extrapolation (2) and without extrapolation indicates that in this problem the use of the extrapolation algorithm reduces the total number of iterations by more than two times (Fig. 4).

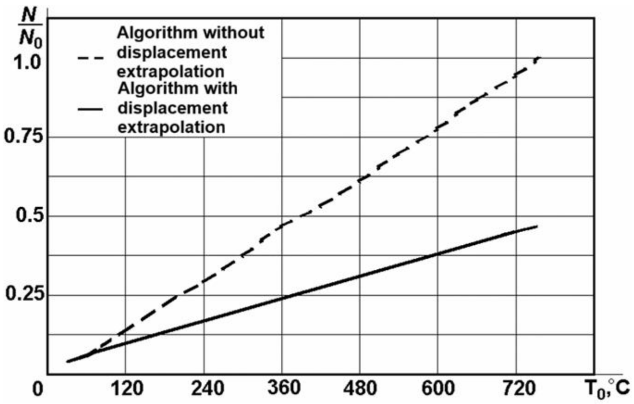


Fig. 4. Comparison of the number of iterations required for solving the problem using algorithms with and without displacement extrapolation

Conclusions. This paper presents a methodology for solving three-dimensional thermo-visco-elasto-plastic problems for prismatic bodies based on the semi-analytical finite element method. A step-by-step algorithm for solving three-dimensional thermo-visco-elasto-plastic problems of prismatic bodies is described. The distinguishing feature of the proposed algorithm is the use of displacement extrapolation for the current step based on the known displacement values from the previous step and information about the variation of external load parameters.

The use of the algorithm with displacement extrapolation makes it possible to obtain reliable results while reducing computational costs by approximately 1.5–3 times, depending on the specific problem considered, which has been demonstrated using a test example.

Application of AI: the AI tool «ChatGPT» was used to translate the article into English.

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НАПІВАНАЛІТИЧНИЙ МЕТОД СКІНЧЕННИХ ЕЛЕМЕНТІВ ДЛЯ РОЗВ'ЯЗАННЯ ПРОСТОРОВИХ ЗАДАЧ ТЕРМОВ'ЯЗКОПРУЖНОПЛАСТИЧНОГО ДЕФОРМУВАННЯ ПРИЗМАТИЧНИХ ТІЛ З УРАХУВАННЯМ ПОШКОДЖЕНОСТІ МАТЕРІАЛУ

У роботі наведено методику розв'язання просторових задач термов'язкопружнопластичності призматичних тіл із використанням напіваналітичного методу скінченних елементів. Описано підхід до моделювання складних за формою неоднорідних конструкцій, на які діють силові та температурні навантаження, що змінюються у просторі й часі. Зазначено, що врахування нелінійних процесів деформування, таких як пластичність і повзучість, вимагає покрокових ітераційних алгоритмів. Запропоновано використання неоднорідних косокутних призматичних скінченних елементів, у формулах яких враховано змінність метричного тензора, що дозволяє суттєво зменшити кількість невідомих та підвищити точність обчислення напружено-деформованого стану. Розроблено алгоритм розв'язання задач термов'язкопружнопластичності, який включає два ітераційні цикли: внутрішній для системи лінійних рівнянь та зовнішній для нелінійних. В основу покладено ітераційну процедуру Ньютона–Канторовича. Особливістю методу є використання екстраполяції переміщень на поточному кроці за результатами попереднього, що забезпечує швидшу збіжність. Наведено приклад моделювання деформування нерівномірно нагрітого куба, для якого результати напіваналітичного методу скінченних елементівпоказали розбіжність менше 5% із відомими даними, підтвердивши достовірність методу. Аналіз свідчить, що максимальні напруження зосереджені в центрі куба, де реалізується стан близький до всебічного стиску, а пластичні деформації знижують рівень напружень до 35%. Застосування алгоритму з екстраполяцією переміщень дозволило зменшити кількість ітерацій більш ніж удвічі та скоротити обчислювальні витрати у 1,5–3 рази. Отримані результати підтверджують ефективність і точність запропонованого підходу для просторових задач термов'язкопружнопластичності складних призматичних тіл.

Ключові слова:напіваналітичний метод скінченних елементів,методика, неоднорідний косокутний призматичний скінченний елемент, метричний тензор.

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SEMI-ANALYTICAL FINITE ELEMENT METHOD FOR SOLVING THREE-DIMENSIONAL PROBLEMS OF THERMO-VISCO-ELASTO-PLASTICITY OF PRISMATIC BODIES WITH MATERIAL DAMAGE CONSIDERATION

The paper presents a methodology for solving three-dimensional problems of thermo-visco-elastoplasticity of prismatic bodies using the semi-analytical finite element method. An approach to modeling geometrically complex heterogeneous structures subjected to spatially and temporally varying mechanical and thermal loads is described. It is noted that accounting for nonlinear deformation processes, such as plasticity and creep, requires step-by-step iterative algorithms. The use of nonhomogeneous skew prismatic finite elements is proposed, in which the formulas account for the variability of the metric tensor. This makes it possible to significantly reduce the number of unknowns and improve the accuracy of approximating the stress–strain state. An algorithm for solving thermo-visco-elastoplastic problems has been developed, which includes two iterative cycles: an inner cycle for solving the system of linear equations and an outer cycle for the nonlinear problem. The approach is based on the Newton–Kantorovich iterative procedure. A distinctive feature of the method is the use of displacement extrapolation at the current step based on the results of the previous step, which ensures faster convergence. An example of modeling the deformation of a non-uniformly heated cube is presented. For this example, the semi-analytical finite element method results showed a discrepancy of less than 5% compared with known reference data, confirming the reliability of the method. The analysis shows that the maximum stresses are concentrated in the center of the cube, where a state close to hydrostatic compression is realized, while plastic deformations reduce the stress level by up to 35%. The application of the algorithm with displacement extrapolation made it possible to reduce the number of iterations by more than half and decrease computational costs by 1.5–3 times. The obtained results confirm the efficiency and accuracy of the proposed approach for three-dimensional thermo-visco-elastoplastic problems of complex prismatic bodies.

Keywords: semi-analytical finite element method, methodology, prismatic inhomogeneous finite element, metric tensor.

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На основі напіваналітичного методу скінченних елементів представлена і апробована методика розв'язання просторових задач термов'язкопружнопластичного деформування призматичних тіл з урахуванням пошкодженості матеріалу.
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Based on the semi-analytical finite element method, a methodology has been developed and validated for solving three-dimensional problems of thermoviscoelastoplastic deformation of prismatic bodies, taking into account material damage.
Fig. 4. Ref. 7.

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