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NON-STATIONARY BENDING VIBRATION OF A HELICOPTER ROTOR BLADE UNDER DISTRIBUTED AERODYNAMIC LOAD

PART 2. FORCED VIBRATIONS OF THE BLADE

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The problem of unsteady forced bending vibrations of a helicopter rotor blade under specified boundary conditions has been solved: one edge of the blade is clamped, the other is free, and a distributed load is applied to the sides of the blade. The helicopter blade is approximated by a thin rectangular plate. A numerical calculation of the problem was performed for a given distribution of transverse load on the front and rear edges of the blade. An analytical solution was found for the problem of forced unsteady bending vibrations of the blade under the acting harmonic load. A numerical calculation of the dimensionless bending of the blade depending on the rotation frequency of the helicopter rotor was performed.
Keywords: helicopter rotor blade, forced bending vibrations of a rectangular plate, distributed load on the blade.

Introduction. In Part 1 of this [1] work, two different approaches, Navier and Levi, were applied, which showed that there is no single solution to the problem of natural vibrations of a cantilever-mounted thin plate. Therefore, the question of correcting the boundary conditions so that the solution is unique arises by itself.

The helicopter blade, which we model with a plate, is actually under a distributed load caused by lift. This physical condition will allow us to find a unique solution. To do this, we will specify the physical boundary conditions and approximate the mathematical model to the real physical problem. Before doing this, let us give a brief analysis of what has been done in this field.

An approach similar to Navier's [2] is proposed in [3] and [4]. It consists in using a two-dimensional Fourier series expansion of the desired deflection of the plate. However, the use of this approach also results in an infinite system of homogeneous algebraic equations (see [4], p. 1302), which does not have a unique solution. The problem of natural vibrations of a cantilever-mounted thin rectangular plate, as discussed in Part 1, cannot be solved analytically, i.e., it is not possible to find a single solution without additional conditions.

Attempts have been made to find approximate solutions for the vibrations of a rectangular plate under certain boundary conditions that differ slightly from the classical boundary conditions. One interesting approach is described in [5]. This approach can be divided into two parts: the first, theoretical part is based on the well-known Levy approach. As a result, we obtain the stationary component $W_n(x,y)$ of the general solution (see [1], equation (25)). Then the unknown constants C_1, C_2, C_3, C_4 are found after introducing 'response' functions, which are formed on the basis of the Fourier series expansion of the bending $W_n(x,y)$ at the boundary. The second part is reduced to solving the boundary equation in terms of bends, which is solved approximately.

In [6], a simplified one-dimensional non-stationary model of coupled bending and torsional vibrations of a helicopter blade is presented. The model is based on a fourth-order differential equation that includes parameters responsible for both bending and torsional vibrations.

In [7], the vibrations of a thin rectangular flexible plate clamped at the boundary and loaded with a sinusoidal force are studied. The problem is solved approximately using the well-known Galerkin's method and polynomial approximation.

Therefore, the analysis described above indicates the need to introduce additional boundary conditions that differ from the classical ones, so that the system of algebraic equations formed after satisfying the boundary conditions becomes compatible, i.e., determines a unique solution to the problem.

Problem statement. Let us set boundary conditions that are somewhat different from the classical ones. Indeed, during rotor rotation, the blade angle of attack constantly changes, the blade seems to float due to the horizontal hinge, i.e. it does not resist the moment $M_x = 0$. However, from the aerodynamics of a helicopter, it is known that the blade surface is under a constant unsteady transverse load $q(x,y,t)$. This feature allows us to specify the boundary conditions that will give a unique solution of the problem.

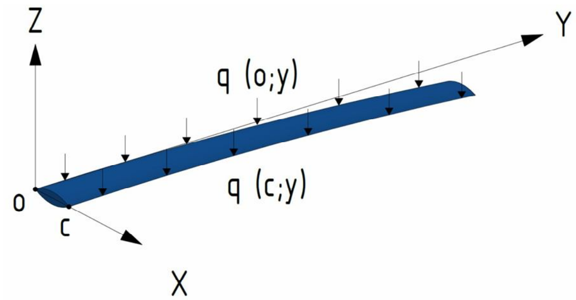

Fig. 1 Taking into account the distributed load on the edges of the blade

Taking into account aerodynamic loads at the blade boundary. For governing equation (see [1], eq.(18)), the boundary conditions of the free edge are physically incorrect, as shown above, since if they are satisfied, we do not have a unique solution to the problem. Let us specify the boundary conditions at $x = 0;c$ as follows: no moment, but transverse forces present at $x = 0;c$ (Fig. 1)**:**

$$M_{x|x=0} = 0\,,\quad M_{x|x=c} = 0\,;\quad Q_{x|x=0} = q(0;y),\quad Q_{x|x=c} = q(c;y)\,. \tag{1}$$

Taking into account the general form for the deflection amplitude (see [1], eq. (25)), the boundary conditions (1) are written:

$$-\left(k_1^{\,2} + \nu\left(\frac{\pi n}{R}\right)^2\right)C_2 + \left(k_2^{\,2} - \nu\left(\frac{\pi n}{R}\right)^2\right)C_4 = 0, \tag{2}$$

$$-\left(k_1^{\,3} + (2-\nu)\left(\frac{\pi n}{R}\right)^2 k_1\right)C_1 \sin\frac{\pi n}{R}y + (k_2^{\,3} - k_2(2-\nu)V^2)C_3 \sin\frac{\pi n}{R}y = q_1(y), \tag{3}$$

$$-\left(\nu\cdot\left(\frac{\pi n}{R}\right)^2 + k_1^{\,2}\right)\sin(k_1\cdot c)C_1 - \left(\nu\cdot\left(\frac{\pi n}{R}\right)^2 + k_1^{\,2}\right)\cos(k_1\cdot c)C_2 +$$

$$+\left(k_2^{\,2} - \nu\cdot\left(\frac{\pi n}{R}\right)^2\right)\mathrm{sh}(k_2\cdot c)C_3 + \left(k_2^{\,2} - \nu\cdot\left(\frac{\pi n}{R}\right)^2\right)\mathrm{ch}(k_2\cdot c)C_4 = 0, \tag{4}$$

$$-\left(k_1^{\,3} + (2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\cos k_1 c\cdot C_1\cdot\sin\frac{\pi n}{R}y - \left(k_1^{\,3} - (2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\sin k_1 c\cdot C_2\cdot\sin\frac{\pi n}{R}y +$$

$$+\left(k_2^{\,3} - (2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\mathrm{ch}k_2 c\cdot C_3\cdot\sin\frac{\pi n}{R}y + C_4\left(k_2^{\,3} - (2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\mathrm{sh}k_2 c\cdot\sin\frac{\pi n}{R}y = q_2(y)\,. \tag{5}$$

Let us multiply equations (3) and (5) by $\sin\frac{\pi n}{R}y$ and perform integration with respect to the variable $0 \le y \le R$. Since

$$\int_0^R \sin^2\frac{\pi n}{R}y\,dy = \frac{R}{2}\,,$$

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then equations (3) and (5) will take the following form after integration:

$$-\left(k_1^{\,3}+(2-\nu)\left(\frac{\pi n}{R}\right)^2 k_1\right)C_1+\left(k_2^{\,3}-(2-\nu)k_2\left(\frac{\pi n}{R}\right)^2\right)C_3=\omega_1, \tag{6}$$

where $\omega_1=\dfrac{2}{R}\int\limits_0^R \sin\dfrac{\pi n}{R}y\cdot q_1(y)dR$.

$$-\left(k_1^{\,3}+(2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\cos k_1 c\cdot C_1-\left(k_1^{\,3}-(2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\sin k_1 c\cdot C_2+$$

$$+\left(k_2^{\,3}-(2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\text{ch}k_2 c\cdot C_3+C_4\left(k_2^{\,3}-(2-\nu)\left(\frac{\pi n}{R}\right)^2\right)\text{sh}k_2 c=\omega_2, \tag{7}$$

where $\omega_2=\dfrac{2}{R}\int\limits_0^R \sin\dfrac{\pi n}{R}y\cdot q_2(y)dR$.

The system of equations (2), (4), (6), (7) is now a non-homogeneous system of algebraic equations, since ω_1,ω_2 are present on its right-hand side. In matrix form, this system of equations can be written as follows:

$$A\cdot\vec{C}=\vec{b}\ , \tag{8}$$

where $\vec{C}=(C_1,C_2,C_3,C_4),\vec{b}=(0,\omega_1,0,\omega_2)$,

$$A=\begin{pmatrix} 0 & -k_1^2-\nu\left(\frac{\pi n}{R}\right)^2 & 0 & k_2^2-\nu\left(\frac{\pi n}{R}\right)^2 \\ -k_1^3-(2-\nu)k_1\left(\frac{\pi n}{R}\right)^2 & 0 & k_2^3-(2-\nu)k_2\left(\frac{\pi n}{R}\right)^2 & 0 \\ -k_1^2\sin k_1 c & -k_1^2\cos k_1 c & k_2^2\text{sh}k_1 c & k_2^2\text{ch}k_2 c \\ -k_1^3\cos k_1 c-(2-\nu)k_1\cos k_1 c\cdot\left(\frac{\pi n}{R}\right)^2 & -k_1^3\sin k_1 c+(2-\nu)k_1\sin k_1 c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\text{ch}k_2 c-(2-\nu)k_2\text{ch}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\text{sh}k_2 c-(2-\nu)k_2\text{sh}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 \end{pmatrix}, \tag{9}$$

Unknown constants for C_1,C_2,C_3,C_4 are found using Cramer's rule:

$$C_1=\frac{\Delta_1}{\Delta},\ C_2=\frac{\Delta_2}{\Delta},\ C_3=\frac{\Delta_3}{\Delta},\ C_4=\frac{\Delta_4}{\Delta}, \tag{10}$$

where $\Delta=\det A$ and

$$\Delta_1=\det\begin{pmatrix} 0 & -k_1^2-\nu\left(\frac{\pi n}{R}\right)^2 & 0 & k_2^2-\nu\left(\frac{\pi n}{R}\right)^2 \\ \omega_1 & 0 & k_2^3-(2-\nu)k_2\left(\frac{\pi n}{R}\right)^2 & 0 \\ 0 & -k_1^2\cos k_1 c & k_2^2\text{sh}k_1 c & k_2^2\text{ch}k_2 c \\ \omega_2 & -k_1^3\sin k_1 c+(2-\nu)k_1\sin k_1 c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\text{ch}k_2 c-(2-\nu)k_2\text{ch}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\text{sh}k_2 c-(2-\nu)k_2\text{sh}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 \end{pmatrix}, \tag{11}$$

$$\Delta_2=\det\begin{pmatrix} 0 & 0 & 0 & k_2^2-\nu\left(\frac{\pi n}{R}\right)^2 \\ -k_1^3-(2-\nu)k_1\left(\frac{\pi n}{R}\right)^2 & \omega_1 & k_2^3-(2-\nu)k_2\left(\frac{\pi n}{R}\right)^2 & 0 \\ -k_1^2\sin k_1 c & 0 & k_2^2\text{sh}k_1 c & k_2^2\text{ch}k_2 c \\ -k_1^3\cos k_1 c-(2-\nu)k_1\cos k_1 c\cdot\left(\frac{\pi n}{R}\right)^2 & \omega_2 & k_2^3\text{ch}k_2 c-(2-\nu)k_2\text{ch}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\text{sh}k_2 c-(2-\nu)k_2\text{sh}k_2 c\cdot\left(\frac{\pi n}{R}\right)^2 \end{pmatrix}, \tag{12}$$

$$\Delta_3=\det\begin{pmatrix} 0 & -k_1^2-\nu\left(\frac{\pi n}{R}\right)^2 & 0 & k_2^2-\nu\left(\frac{\pi n}{R}\right)^2 \\ -k_1^3-(2-\nu)k_1\left(\frac{\pi n}{R}\right)^2 & 0 & \omega_1 & 0 \\ -k_1^2\sin k_1c & -k_1^2\cos k_1c & 0 & k_2^2\mathrm{ch}k_2c \\ -k_1^3\cos k_1c-(2-\nu)k_1\cos k_1c\cdot\left(\frac{\pi n}{R}\right)^2 & -k_1^3\sin k_1c+(2-\nu)k_1\sin k_1c\cdot\left(\frac{\pi n}{R}\right)^2 & \omega_2 & k_2^3\mathrm{sh}k_2c-(2-\nu)k_2\mathrm{sh}k_2c\cdot\left(\frac{\pi n}{R}\right)^2 \end{pmatrix}, \quad (13)$$

$$\Delta_4=\det\begin{pmatrix} 0 & -k_1^2-\nu\left(\frac{\pi n}{R}\right)^2 & 0 & 0 \\ -k_1^3-(2-\nu)k_1\left(\frac{\pi n}{R}\right)^2 & 0 & k_2^3-(2-\nu)k_2\left(\frac{\pi n}{R}\right)^2 & \omega_1 \\ -k_1^2\sin k_1c & -k_1^2\cos k_1c & k_2^2\mathrm{sh}k_1c & 0 \\ -k_1^3\cos k_1c-(2-\nu)k_1\cos k_1c\cdot\left(\frac{\pi n}{R}\right)^2 & -k_1^3\sin k_1c+(2-\nu)k_1\sin k_1c\cdot\left(\frac{\pi n}{R}\right)^2 & k_2^3\mathrm{ch}k_2c-(2-\nu)k_2\mathrm{ch}k_2c\cdot\left(\frac{\pi n}{R}\right)^2 & \omega_2 \end{pmatrix}. \quad (14)$$

As C_i depend on the frequency of the external load ω, the number n, the chord length c of the blade, the radius R of the blade, the thickness h of the blade, the density ρ of the material, and the cylindrical stiffness D.

Forced vibrations of the plate - helicopter rotor blades. Let the external load be harmonic $\cos\omega t$ (Fig. 2), then equation Lagrange-Sophie Germain (see [1], eq.(1)) will take the form:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\ddot{w}=q(x,y)\cos\omega t\,. \quad (15)$$

We will solve this equation using the principal coordinates method $\eta_n(t)$ [8]. According to this method, the general solution will be presented as follows:

$$w(x,y,t)=\sum_{i=1}^{n}W_n(x,y)\eta_n(t)\,, \quad (16)$$

where

$$W_n(x,y)=(C_1\sin k_1x+C_2\cos k_1x+C_3\mathrm{sh}k_2x+C_4\mathrm{ch}k_2x)\cdot\sin\frac{\pi n}{R}y\,.$$

After substituting (16) into (15), we obtain the following equation:

$$D\sum_{n=1}^{\infty}\left[X_n(x)\sin\frac{\pi n}{R}y-2X_n(x)''\left(\frac{\pi n}{R}\right)^2\sin\frac{\pi n}{R}y+X_n(x)\left(\frac{\pi n}{R}\right)^4\sin\frac{\pi n}{R}y\right]\eta_n(t)+$$

$$+\rho h\sum_{n=1}^{\infty}X_n(x)\sin\frac{\pi n}{R}y\ddot{\eta}_n(t)=q(x,y)\cos\omega t. \quad (17)$$

Next, according to the method of principal coordinates, we multiply both parts (17) by $\sin\frac{\pi n}{R}y$ and perform integration over the variable x from 0 to c, and over the variable y from 0 to R. As a result, we have:

$$DM_n\frac{R}{2}\eta_n(t)+\rho h\frac{R}{2}N_n\ddot{\eta}_n(t)=q_n(t)\,, \quad (18)$$

where

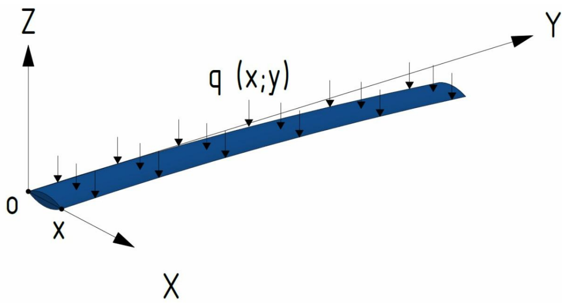


Fig. 2 Forced vibrations of the blade under the action of aerodynamic load

$$M_n = \int_0^c \left\{ (k_1^4 - 2k_1^2 \left(\frac{\pi n}{R}\right)^2 + \left(\frac{\pi n}{R}\right)^4)(C_1 \sin k_1 x + C_2 \cos k_1 x) \right\} dx +$$

$$+ \int_0^c \left\{ (k_2^4 - 2k_2^2 \left(\frac{\pi n}{R}\right)^2 + \left(\frac{\pi n}{R}\right)^4)(C_3 \mathrm{sh} k_2 x + C_4 \mathrm{ch} k_2 x) \right\} dx, \quad (19)$$

$$N_n = \int_0^c (C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 \mathrm{sh} k_2 x + C_4 \mathrm{ch} k_2 x) dx, \quad (20)$$

$$q_n(t) = (\int_0^c dx \int_0^R q(x,y) \sin \frac{\pi n}{R} y dy) \cdot \cos \omega t. \quad (21)$$

Let us rewrite equation (18) in a compact form:

$$\ddot{\eta}_n(t) + n_1 \cdot \eta_n(t) = n_2 \cos \omega t, \quad (22)$$

where

$$n_1 = \frac{D}{\rho h} \cdot \frac{M_n}{N_n}, n_2 = \frac{2}{R \rho h N_n} \cdot \int_0^c dx \int_0^R q(x,y) \sin \frac{\pi n}{R} y dy. \quad (23)$$

The differential equation (22) is a linear differential equation of the second order. The characteristic equation for it gives the roots $\pm i\sqrt{n_1}$ or $\pm\sqrt{n_1}$. The first case is realized if $M_n / N_n > 0$. Let us consider it, then:

$$\eta_n(t) = A_n \sin \sqrt{n_1} t + B_n \cos \sqrt{n_1} t + \frac{n_2}{n_1 - \omega^2} \cos \omega t. \quad (24)$$

The general of equation (15) in the non-resonant region will take the form:

$$w(x,y,t) = \sum_{i=1}^{n} W_n(x,y)(A_n \sin \sqrt{n_1} t + B_n \cos \sqrt{n_1} t + \frac{n_2}{n_1 - \omega^2} \cos \omega t). \quad (25)$$

Under zero initial conditions $\dot{\eta}_{t=0}(t) = 0,\ \eta_{t=0}(t) = 0$, we have:

$$w(x,y,t) = \begin{cases} \sum_{i=1}^{n} \frac{n_2}{n_1 - \omega^2} (C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 \mathrm{sh} k_2 x + C_4 \mathrm{ch} k_2 x) \cdot \sin \frac{\pi n}{R} y \cdot (\cos \omega t - \cos \sqrt{n_1} t), & \omega \neq \sqrt{n_1} \\ \sum_{i=1}^{n} \frac{n_2}{2\sqrt{n_1}} (C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 \mathrm{sh} k_2 x + C_4 \mathrm{ch} k_2 x) \cdot \sin \frac{\pi n}{R} y \cdot t \sin \sqrt{n_1} t, & \omega \approx \sqrt{n_1}. \end{cases} \quad (26)$$

In solution (26), the non-resonant ($\omega \neq \sqrt{n_1}$) and resonant ($\omega \approx \sqrt{n_1}$) zones of plate vibration are separated. For the resonant region, L'Hôpital's rule (differentiation by variable ω and limit transition at $\omega \to \sqrt{n_1}$) is applied by analogy with (see [8], p. 423). Solution (26) is somewhat similar to the solution given in [9]. However, it differs in that we did not equate the zero determinant of the matrix because we were looking for a unique solution to the problem. That is, we did not establish an additional dependence between the eigenvalues. Therefore, the coefficients C_1, C_2, C_3, C_4 in solution (26) differ from the similarly designated coefficients of the problem in [9].

Example of calculating the forced vibration problem for a given distribution $q(y)$ at the front and rear edges of the blade. During flight, excess pressure occurs under the helicopter blade, which creates lift L. If the blade has no geometric twist, the lift distribution varies along the blade span by coordinate y. It depends on the blade flow velocity and cross-sectional shape, which vary along the blade span. In the case of the above problem, we do not yet take into account the shape of a thin blade in cross section. The study of the influence of shape on the distribution of lift is planned in further research. In this work, we will only take into account the change in speed along the blade span. We will assume that the circulation is $\Gamma = \mathrm{const}$.

From helicopter aerodynamics, it is known that the lift force L of a blade section located at a distance r along the blade span varies as follows [10], [11]:

$$dL / dr = \rho \omega r \Gamma \, , \tag{27}$$

or after integration,

$$L(r) = \frac{\rho \omega \Gamma}{2} r^2 . \tag{28}$$

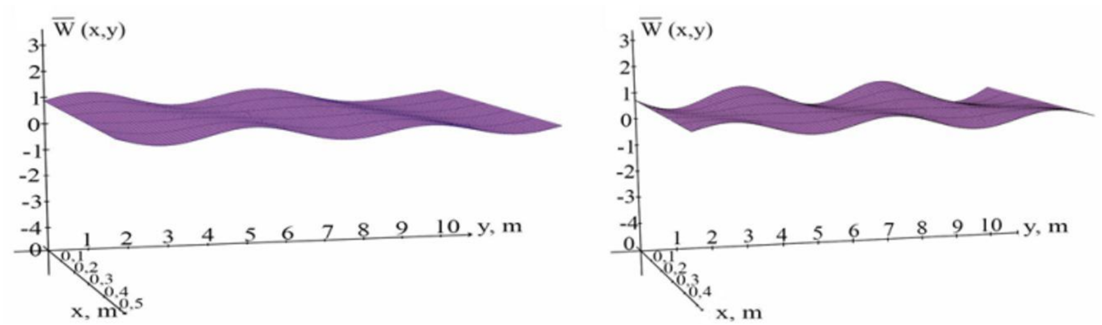
Under the condition $\Gamma = \text{const}$ we have a quadratic distribution $L(r)$. It is also known that the end vortices on the blade reduce the lift force, i.e., the load on the blade, to zero. This property is approximately reflected in the following mathematical distribution from [12]:

$$q(y) = \frac{y^2 \sqrt[3]{(y-R)^2}}{(y-R+1)^2 + 1} \, , \tag{29}$$

where in our case $y \equiv r$. Therefore, after determining the coefficients n_1, n_2 using formulas (19), (20), (29) and substituting them into equation (26), we will obtain the distribution of the helicopter blade deflection.

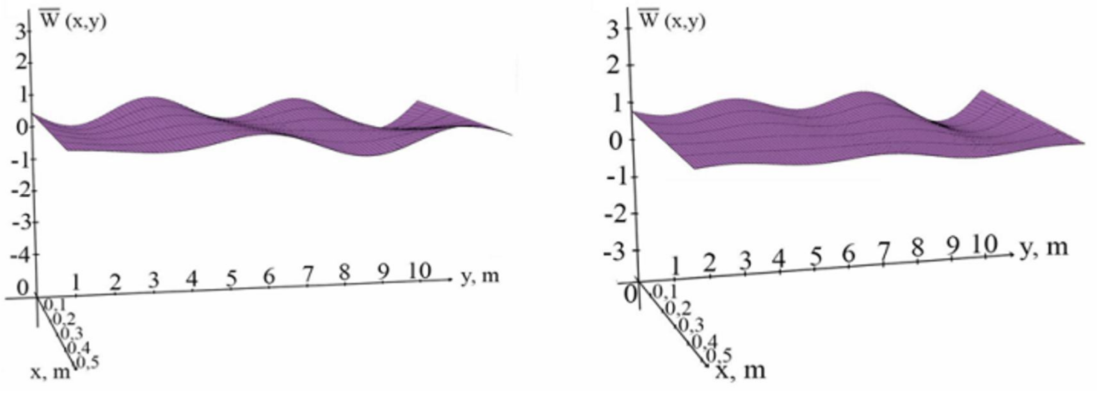
We are interested in the dependence of the blade deflection on the rotor rotation frequency ω at given parameters $c = 0{,}5m,\ R = 10m,\ h = 0{,}05m$.

Fig. 3-5 shows the calculations of the dimensionless blade bending amplitude $\overline{W}(x, y)$. They reflect the wave nature of the blade's natural vibrations at different values of revolutions per second: $n = 30; 40; 45; 50; 55$. The specified number of revolutions per second corresponds to the standard range of helicopter rotor rotation frequencies - $2000 - 3000$ rpm. A characteristic feature common to all the above calculations is an increase in the amplitude of vibrations at the blade tip with an increase in the helicopter rotor rotation frequency.



(a) (b)

Fig. 3 Distribution of dimensionless blade bending: (a) $\omega = 2\pi \cdot 35$; (b) $\omega = 2\pi \cdot 40$



(a) (b)

Fig. 4. Distribution of dimensionless blade bending: (a) $\omega = 2\pi \cdot 45$; (b) $\omega = 2\pi \cdot 50$

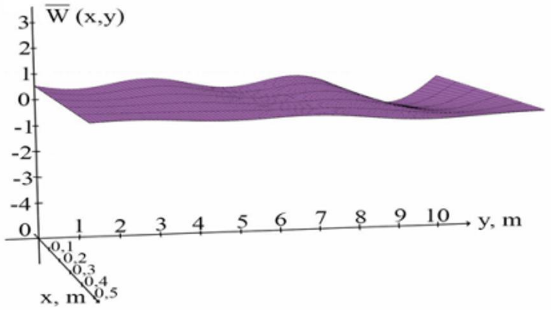


Fig. 5. Distribution of dimensionless blade bending: $\omega = 2\pi \cdot 55$

Note. Non-stationary calculations according to expression (26) were not performed in this work. The non-stationarity of vibrations should lead to additional bending. The solution to this problem is planned for future research, where the complete non-stationary load distribution $q(x,y,t)$ and the blade cross-section shape corresponding to the actual conditions of the flow around the blade will be taken into account.

Conclusions. This paper presents and analytically solves the problem of small bending vibrations of a rectangular thin plate, a helicopter blade, with one edge cantilevered, the opposite edge free, and a distributed load on the other two edges. The rest of the blade is free and unloaded. The solution is found using the Levy approach. Further, using the normal coordinate method, an analytical solution is found for the problem of forced vibrations of the plate, taking into account the force distribution specified on the surface. The calculation of the dimensionless distribution of the bending amplitudes of the plate (rotor blade) showed the wave nature of the bending vibrations, as well as the dependence of the amplitude of the blade tip vibrations on the rotation frequency of the helicopter blade.

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НЕСТАЦІОНАРНІ ЗГИНАЛЬНІ КОЛИВАНННЯ ЛОПАТІ РОТОРА ВЕРТОЛЬОТА ПІД ДІЄЮ РОЗПОДІЛЕНОГО АЕРОДИНАМІЧНОГО НАВАНТАЖЕННЯ. ЧАСТИНА 2. ВИМУШЕНІ КОЛИВАННЯ ЛОПАТІ

Кожна технічна постановка задачі потребує у подальшому адекватної фізичної моделі. Ця модель повинна враховувати основні фізичні властивості, наблизити фізичну постановку задачі безпосередньо до технічної задачі. У Частині 1 даної роботи наведено експериментальні миттєві фотографії згинальних коливань лопаті ротора вертольота, які схожі з малими згинальними коливаннями консольно защемленої тонкої пластини. Відомо, що лопаті вертольота виготовляються жорсткими. Тому для моделювання малих згинальних коливань в даній роботі застосовано гіпотези Кірхгова - Лява, рівняння Лагранжа - Софі Жермен.

На сьогодні існуючі підходи розв'язання задачі про власні згинальні коливання тонкої прямокутної пластини за умови консольного защемлення призводять до невизначеності, не єдиності розв'язку задачі. Тому, у Частині 2 даної роботи з метою отримання єдиного розв'язку задачі, на двох краях пластини задається ненульовий розподіл нормальних зусиль. Ці зусилля ураховані лише у граничних умовах, а решта поверхні лопаті є вільною. Тому у праву

частину рівняння Лангранжа-Софі Жермен ніяких сил не додається. Таке уточнення граничних умов наближає фізичну постановку задачі саме до реальної технічної задачі: на лопать під час її обертання діють сили, що спричиняють підйомну силу, яка обумовлює нормальне до поверхні лопаті розподілене навантаження.

Зміна граничних умов на двох краях лопаті призвела до того, що система лінійних алгебраїчних рівнянь з однорідної перетворилась на неоднорідну. Це дозволило на основі правила Крамера знайти єдиний розв'язок задачі, однозначно визначити шукані константи інтегрування. Оскільки рух лопаті є періодичним, то задачу розв'язано за умови гармонічного у часі навантаження пластини.

Для отримання аналітичного розв'язку задачі для вимушених коливань застосовано метод нормальних координат. Він дозволяє використати розв'язок задачі для власних коливань, стаціонарну частину його, та на його основі отримати рівняння для нормальної координати, що описує нестаціонарність загального розв'язку задачі. Скориставшись правилом Лопіталя, отриманий загальний розв'язок розділено по частотам на нерезонансну та резонансну область.

Як приклад чисельного розрахунку, розраховано амплітуди прогину власних коливань пластини, лопаті вертольота, для різних частот її обертання. Виявлено, що амплітуда прогину має хвильовий характер, дещо збільшуються до зовнішнього кінця лопаті зі збільшенням частоти обертання ротора вертольота.

Ключові слова: лопать ротора вертольота, власні та вимушені згинальні коливання прямокутної пластини, аналітичний розв'язок.

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NON-STATIONARY BENDING VIBRATION OF A HELICOPTER ROTOR BLADE UNDER DISTRIBUTED AERODYNAMIC LOAD. PART 2. FORCED VIBRATIONS OF THE BLADE

Each technical problem statement requires an adequate physical model. This model must take into account the basic physical properties and bring the physical problem statement closer to the technical problem. Part 1 of this work presents experimental instantaneous photographs of the bending vibrations of a helicopter rotor blade, which are similar to the small bending vibrations of a cantilever-clamped thin plate. It is known that helicopter blades are made rigid. Therefore, to model small bending vibrations, we apply Kirchhoff-Love's hypothesis and Lagrange-Sophie Germain's equation in this work.

Currently, existing approaches to solving the problem of natural bending vibrations of a thin rectangular plate under cantilever clamping conditions lead to uncertainty and non-uniqueness of the solution. Therefore, in Part 2 of this work, in order to obtain a unique solution to the problem, a non-zero distribution of normal forces is set at the two edges of the plate. These forces are taken into account only in the boundary conditions, and the rest of the blade surface is free. Therefore, no forces are added to the right-hand side of the Langrange-Sophie Germain equation. This refinement of the boundary conditions brings the physical formulation of the problem closer to the real technical problem: during rotation, forces act on the blade, causing a lifting force, which determines the load distributed normal to the blade surface.

The change in boundary conditions at the two edges of the blade resulted in the system of linear algebraic equations changing from homogeneous to non-homogeneous. This made it possible to find a unique solution to the problem based on Cramer's rule and to unambiguously determine the desired integration constants. Since the blade motion is periodic, the problem is solved under the condition of a time-harmonic load on the plate.

To obtain an analytical solution of the problem for forced vibrations, the normal coordinate method is used. It allows using the solution of the problem for natural vibrations, its stationary part, and on its basis, obtaining an equation for the normal coordinate that describes the non-stationarity of the general solution of the problem. Using L'Hôpital's rule, the obtained general solution is divided by frequency into non-resonant and resonant regions.

As an example of numerical calculation, the amplitudes of the bending of the natural vibrations of a plate, a helicopter blade, are calculated for different frequencies of its rotation. It was found that the amplitude of the bending has a wave character, slightly increasing towards the outer end of the blade with an increase in the rotation frequency of the helicopter rotor.

Keywords: helicopter rotor blade, natural and forced bending vibrations of a rectangular plate, analytical solution.

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Отримано точний аналітичний розв'язок задачі про власні та вимушені коливання пластини на основі рівняння Лагранжа - Софі Жермен за умов консольного закріплення одного кінця, вільного протилежного кінця лопаті та розподіленого навантаження на бічних сторонах пластини, лопаті вертольота. У якості приклада виконано чисельний розрахунок модельної задачі для гармонічного розподілу в часі навантаження. Нестаціонарна задача розв'язана за допомогою методу нормальних координат.

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The problem of non-stationary bending vibrations of a helicopter rotor blade has been solved. The helicopter blade is approximated by a thin rectangular plate. The natural and forced vibrations of the plate are investigated based on the Lagrange-Sophie Germen equation under the conditions of cantilever fixation of one end and three free ends. It is shown that the solution is not unique. Under corrected boundary conditions on two sides of the plate, an analytical solution to the problem of free and forced vibrations of the plate is obtained. As an example, a numerical calculation of the model problem under a specific load is performed.

Fig. 5. Ref. 12.

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