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FREE VIBRATIONS OF LAYERED ANISOTROPIC THICK-WALLED CYLINDRICAL SHELLS

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In this work an approach is proposed to solve the problem of free vibrations of layered thick-walled cylindrical shells made of an anisotropic material, the elastic characteristics of which are in the same plane, tangent to the middle surface. A three-dimensional system of homogeneous differential equations of motion of the linear theory of elasticity of an anisotropic body on the basis of the modified Ky-Washizu variational principle was developed. It was recorded in the cylindrical coordinate system for the appropriate boundary conditions on the surfaces and ends of the shell. Using the analytical Bubnov – Galerkin method to reduce the dimension of a three-dimensional system, an approach to obtaining an infinite one-dimensional system of differential equations is presented. It gives possible to determine the frequencies of free vibrations of thick-walled unsymmetric laminate anisotropic cylindrical shell structures. Based on the developed approach to the calculation of free vibrations in the spatial setting of a thick-walled anisotropic cylindrical shell, an analysis of the results of frequency determination was carried out. The proposed approach significantly expands the possibilities of calculating shell structures from composite materials.

Key words: cylindrical anisotropic shells, free vibrations, three-dimensional system of equations of motion, Ky-Washizu variational principle, Bubnov–Galerkin method.

Introduction

It is generally known that dynamic calculations of shell structures depend, among other things, on the determination of their free vibration parameters. This is due to the requirements of practical needs in various fields of mechanical engineering, instrument engineering, aviation and aerospace engineering, construction, etc. The application of classical and refined theories in the study of shells has made it possible to obtain approximate solutions to such problems. Paper [1] presents methods for solving the problems of calculating the natural frequencies and corresponding vibration forms of elastic shells when using isotropic, orthotropic, and one-plane elastic symmetry material models. The effectiveness of the exact reduction of a two-dimensional eigenvalues problem to a sequence of separate one-dimensional problems with their subsequent integration by the method of discrete orthogonalization is proved. In the first chapter [2], using the theoretical principles [1], calculations are given to determine the parameters of natural vibrations of anisotropic plates and a thin anisotropic cylindrical shell. It is shown that a seven-layered anisotropic shell can be calculated with sufficient accuracy as an orthotropic one. In the same paper, the author considers axisymmetric vibrations of cylindrical shells with one surface of elastic property symmetry. The authors in their study [3]consider approaches to solving linear and non-linear problems of shell mechanics based on discrete-continuum methods in the classical, refined, and spatial models for isotropic and orthotropic inhomogeneous shells with variable geometric and mechanical parameters. The authors present results of their studies of the dynamic characteristics of shells of various shapes and end fixation.

Paper [4] present two approaches to the calculation of closed thick layered anisotropic cylindrical shells based on the division of a cylindrical shell by thickness into a number of composite cylindrical shells. Having satisfied the contact conditions on the surfaces between them, the authors determine the natural frequencies of bending vibrations of the original shell. In the first approach, the distribution of functions by thickness is established on the basis of an analytical solution of the corresponding system of differential equations. In the second approach, they are determined by approximation with polynomial functions. In the paper [5] the authors use the theoretical foundations [4] to determine the frequencies of free vibrations, and for their implementation they use a semi-analytical finite element method in addition to the successive narrowing of the search interval method. An analysis of the behavior of shells under free and forced vibrations was carried out.

The paper [6] proposes an approach to determining the frequencies and shapes of free vibrations of systems composed of rotating shells of different geometry and relative thickness, continuously and/or discretely inhomogeneous in thickness, made of isotropic, orthotropic and anisotropic materials with one plane of elastic symmetry. The approach includes the creation of a mathematical model of vibrations based on the classical Kirchhoff-Love theory, the refined Tymoshenko-type theory, and the spatial theory of elasticity (partial case). Numerical-analytical approach to solving the corresponding two-dimensional (three-dimensional) problems involves reducing their dimensionality and using the methods of successive approximations and stepwise search along with the method of discrete orthogonalisation. In the paper [7] the dynamic characteristics of a thick-walled steel cylindrical shell were determined using the finite element method. A comparative analysis of the calculated frequencies and forms of free vibrations with those obtained experimentally was carried out. The frequency coefficients obtained by the authors indicate the dependence of the natural frequency on the material characteristics. In the paper [8], the authors determined the frequencies and shapes of free vibrations of a thick open steel cylindrical shell of elliptical cross-section for different variants of end fixation and physical and mechanical parameters using the finite element method.

The authors of the paper [9] state that the introduction of the concept of variable stiffness has expanded the scope of high-performance lightweight composite structures use. They consider dynamic excitation of prestressed aerospace structures allowing for more efficient solutions with higher overall stiffness and fundamental natural frequency. In this context, the Ritz method is used to analyze the natural frequencies of prestressed multilayer plates and open variable-stiffness shells. The first-order shear deformation theory is considered without further assumptions on the structure thinness. The parametric studies demonstrate the flexibility provided by the variable stiffness concept in finding compromise solutions for the analysis of natural vibration frequencies.

The use of modern composite materials and structural solutions leads to an increase in the requirements for the construction of mathematical models of vibrations that would fully satisfy practical use in terms of accuracy[10 - 13]. It is known that the development of such models for solving, for example, dynamic problems of thick-walled anisotropic cylindrical shells is possible only within the framework of the spatial theory of elasticity.

The anisotropy of shell structures considered in the work is determined by the mismatch of the main elastic directions between the proper axes of the orthotropic material and the curvilinear coordinate system of the shells (Fig. 1). This occurs, for instance, when manufacturing anisotropic cylindrical shells by winding the fibers of the original orthotropic material onto a mandrel. In this case, the resulting shell material is considered to have a single plane of elastic symmetry tangent to the shell middle surface[1, 2, 10, 14 -16].

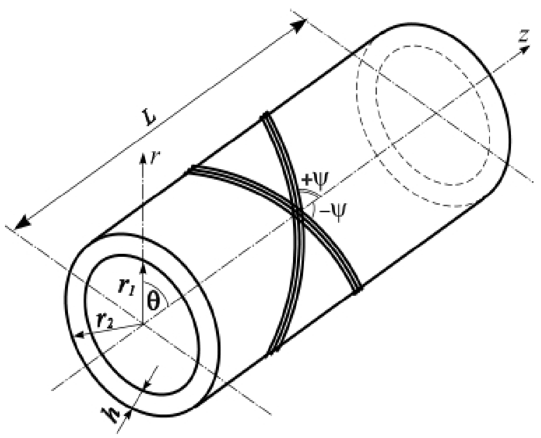

Fig. 1. Thick-walled anisotropic cylindrical shell

This paper presents an approach to obtaining three-dimensional differential equations of elasticity theory to determine the free vibration frequencies of an anisotropic body based on the functionality modification of the generalized Ky-Washizu principle. Assuming that the anisotropic body is a composite thick-walled hollow cylinder (Fig. 1), we solve the obtained system of differential equations by combining the following methods in the computational process: analytical Bubnov – Galerkin and numerical discrete orthogonalization.

1. Materials and Methods

1.1. Basic equations. Ky-Washizu's variational principle

In accordance with the Ky-Washizu variational principle [17], the equations of motion, elasticity relations (equations of state), geometric relations, and the appropriate boundary conditions can be obtained from the stationarity condition of the functional Π_1 which is determined from the integral for dynamic problems:

$$\Pi_1=\int_{t_1}^{t_2}\left\{\iiint_V\left\{W\left(e_{ij}\right)-T\left(u_i\right)+\Phi\left(u_i\right)-\sigma_{ij}\left[e_{ij}-\frac{1}{2}\left(u_{i;j}+u_{j;i}\right)\right]\right\}dV+\iint_{S_1}\Psi\left(u_i\right)dS-\iint_{S_2}p_i\left(u_i-\bar{u}_i\right)dS\right\}dt\,. \quad (1.1)$$

The displacements u_i, deformations e_{ij}, stresses σ_{ij} and stresses p_i on the surface S_2 caused by the displacements $\bar{u}_i$ vary without additional conditions. Also, in this functional $W(e_i)$ – is the potential strain energy, $T(u_i)$ – kinetic energy, $\Phi(u_i)$, $\Psi(u_i)$ – are the volume and surface load potentials, u_i – are components of the displacement eigenvector, a semicolon before the parameters i, j denotes the covariant derivative in the coordinate with the corresponding index i, j = 1, 2, 3. The potential strain energy in the eigenvector-matrix representation is written as follows:

$$W(e_{ij})=\frac{1}{2}\varepsilon^T B\varepsilon\,, \quad (1.2)$$

where $\varepsilon^T=(\varepsilon_{zz},\varepsilon_{\theta\theta},\varepsilon_{rr},2\varepsilon_{r\theta},2\varepsilon_{rz},2\varepsilon_{z\theta})$, B – is the matrix of elasticity coefficients.

If we introduce a eigenvector $\sigma^T=\left(\sigma_{zz},\ \sigma_{\theta\theta},\ \sigma_{rr},\ \sigma_{r\theta},\ \sigma_{rz},\ \sigma_{z\theta}\right)$, then from the stationarity condition $\delta\Pi_1=0$ we obtain the following equations:

$$\sigma=B\varepsilon\,, \quad (1.3)$$

$$\varepsilon=\varepsilon(u)\,, \quad (1.4)$$

$$\sigma_{ij;j}+f_i=0\,, \quad (1.5)$$

as well as boundary conditions $\sigma_{ij}n_j=\bar{F}_i$ on the surface S_1 and displacements $u_i=\bar{u}_i$ and stresses $p_i=\sigma_{ij}n_j$ on S_2.

The deformation relationship (1.4) show the relationship between deformations and displacements. Inverse of the elasticity relations (1.3), the strain-stress dependence is represented as:

$$\varepsilon=A\sigma\,, \quad (1.6)$$

where matrix $A=B^{-1}$.

The coefficients of the matrix A are denoted by a_{ij} and the matrices B – b_{ij} $(i,\,j=\overline{1,6})$. The matrices A and B – are symmetric, since $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$. In the following, we also establish the relationship between the matrices A and B.

1.2. Modified mixed variational principle

Letus suggest that [10, 15, 18, 19, 20] derive the mixed variational principle modified to Ky-Washizu and separate the eigenvectors σ and ε into two parts so that:

$$\sigma_1^T=\left(\sigma_{rr},\tau_{r\theta},\tau_{rz}\right),\ \sigma_2^T=\left(\sigma_{zz},\sigma_{\theta\theta},\tau_{z\theta}\right),\ \varepsilon_1^T=\left(\varepsilon_{rr},\varepsilon_{r\theta},\varepsilon_{rz}\right),\ \varepsilon_2^T=\left(\varepsilon_{zz},\varepsilon_{\theta\theta},\varepsilon_{z\theta}\right). \quad (1.7)$$

To shorten elasticity relation record (1.3), it will be expressed in matrix form:

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}, \quad (1.8)$$

where for the blocks A_{ij} according to the accepted division (1.7), from the matrix A in (1.6) for an anisotropic material whose elastic properties are in the same plane, we obtain:

$$A_{11}=\begin{bmatrix}a_{33} & 0 & 0\\ 0 & a_{44} & a_{45}\\ 0 & a_{45} & a_{55}\end{bmatrix},\ A_{12}=\begin{bmatrix}a_{31} & a_{32} & a_{36}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ A_{21}=\begin{bmatrix}a_{13} & 0 & 0\\ a_{23} & 0 & 0\\ a_{36} & 0 & 0\end{bmatrix},\ A_{22}=\begin{bmatrix}a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}. \quad (1.9)$$

From the matrix dependence $\varepsilon=A\sigma$ using (1.8), we obtain:

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}\sigma_2 \ , \tag{1.10}$$

$$\varepsilon_2 = A_{21}\sigma_1 + A_{22}\sigma_2 \ , \tag{1.11}$$

then from (1.11) we have an expression for:

$$\sigma_2 = A_{22}^{-1}\varepsilon_2 - A_{22}^{T}A_{12}^{T}\sigma_1 \ . \tag{1.12}$$

We substitute the latter into (1.10) and then:

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}A_{22}^{-1}\varepsilon_2 - A_{12}A_{22}^{-1}A_{21}\sigma_1 = A_{12}A_{22}^{-1}\varepsilon_2 + \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)\sigma_1 \ . \tag{1.13}$$

From (1.13) we find σ_1 :

$$\sigma_1 = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \cdot \varepsilon_1 - \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \cdot A_{12}A_{22}^{-1}\varepsilon_2 \ . \tag{1.14}$$

From the matrix dependence:

$$\sigma = B \cdot \varepsilon \tag{1.15}$$

Let us write down:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} . \tag{1.16}$$

For the considered type of anisotropic material, we have:

$$\sigma_1 = B_{11} \cdot \varepsilon_1 + B_{12} \cdot \varepsilon_2 , \tag{1.17}$$

$$\sigma_2 = B_{21} \cdot \varepsilon_1 + B_{22} \cdot \varepsilon_2 . \tag{1.18}$$

By comparing (1.15) and (1.14), we establish the relationship between the matrices:

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} , \tag{1.19}$$

$$B_{12} = -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} A_{12}A_{22}^{-1} . \tag{1.20}$$

In the expression for σ_2 (1.12) we substitute (1.14) and then we obtain:

$$\begin{aligned} \sigma_2 &= A_{22}^{-1}\varepsilon_2 - A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\varepsilon_1 + A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1} - A_{21})A_{12}A_{22}^{-1}\varepsilon_2 = \\ &= -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\varepsilon_1 + \left[A_{22}^{-1} + A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})A_{12}A_{22}^{-1}\right]\varepsilon_2 . \end{aligned} \tag{1.21}$$

In regard to (1.21) and (1.18), we have the following relationship between the matrices:

$$B_{21} = -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} , \tag{1.22}$$

$$B_{22} = A_{22}^{-1} + A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} A_{12}A_{22}^{-1} . \tag{1.23}$$

Thus, expressions (1.19) and (1.20) and (1.22) and (1.23) establish the relationship between matrices in the two forms of the generalized Hooke's law for an anisotropic material (1.3) and (1.6), whose elastic properties are determined by a single plane of symmetry of the elastic characteristics.

In contrast to the Ky-Washizu principle, we assume that the displacements u_r, u_θ, u_z, deformations ε_{zz}, $\varepsilon_{z\theta}$, $\varepsilon_{\theta\theta}$ and stresses σ_{rr}, $\tau_{r\theta}$, τ_{rz} in the functional (1.1) are independent. From the equations:

$$\sigma_1 = B_{11} \cdot \varepsilon_1 + B_{12} \cdot \varepsilon_2 , \tag{1.24}$$

$$\varepsilon_2 = A_{21} \cdot \sigma_1 + A_{22} \cdot \sigma_2 \tag{1.25}$$

we find:

$$\varepsilon_1 = B_{11}^{-1} \cdot \sigma_1 - B_{11}^{-1}B_{12} \cdot \varepsilon_2 , \sigma_2 = A_{22}^{-1} \cdot \varepsilon_2 - A_{22}^{-1}A_{21} \cdot \sigma_1 . \tag{1.26}$$

The expression for the potential $W(e_{ij})$ with new notations will be given the following form:

$$W(\varepsilon) = \frac{1}{2}\left(\varepsilon_1^T B_{11}\varepsilon_1 + \varepsilon_1^T B_{12}\varepsilon_2 + \varepsilon_2^T B_{12}^T\varepsilon_1 + \varepsilon_2^T B_{22}\varepsilon_2\right) . \tag{1.27}$$

From expression (1.27), taking into account (1.26), we exclude ε_1. Then we get:

$$\begin{aligned} W(\sigma_1, \varepsilon_2) = \frac{1}{2}\Big[&(B_{11}^{-1}\sigma_1 - B_{11}^{-1}B_{12\varepsilon_2})^T B_{11}(B_{11}^{-1}\sigma_1 - B_{11}^{-1}B_{12}\varepsilon_2) + \\ &+ (B_{11}^{-1}\sigma_1 - B_{11}^{-1}B_{12}\varepsilon_2)^T B_{12}\varepsilon_2 + \varepsilon_2^T B_{12}^T(B_{11}^{-1}\sigma_1 - B_{11}^{-1}B_{12}\varepsilon_2) + \varepsilon_2^T B_{22}\varepsilon_2\Big] . \end{aligned}$$

After simple transformations, we'll get the final result:

$$W(\sigma_1,\varepsilon_2)=\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1+\frac{1}{2}\varepsilon_2^T\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2 \ . \tag{1.28}$$

Similarly, we transform the expression $\sigma_{ij}\,\varepsilon_{ij}$. After comparing the matrix expressions:

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}\sigma_2 \text{ та } \sigma_1 = B_{11}\cdot\varepsilon_1 + B_{12}\cdot\varepsilon_2$$

it is easy to establish that $B_{12}B_{22}^{-1} = -A_{11}^{-1}A_{12}$.

Then,

$$\sigma_{ij}\,\varepsilon_{ij}=\sigma_1^T B_{11}^{-1}\sigma_1+\varepsilon_2^T\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2 \ . \tag{1.29}$$

Excluding the stress eigenvector component σ_2 from the expression $\sigma_{ij}\,\varepsilon_{ij}(u)$, we obtain

$$\sigma_{ij}\varepsilon_{ij}(u)=\left(\varepsilon_1^T(u)+\varepsilon_2^T(u)B_{12}^T B_{11}^{-1}\right)\sigma_1+\varepsilon_2^T(u)\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2 \ . \tag{1.30}$$

Assembling expressions (1.28 - 1.30), we write down the potential:

$$W_1=W(\sigma_1,\varepsilon_2)-\sigma_{ij}\left(\varepsilon_{ij}-\varepsilon_{ij}(u)\right)=-\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1-\frac{1}{2}\varepsilon_2^T\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2+$$

$$+\left(\varepsilon_1^T(u)+\varepsilon_2^T(u)B_{12}^T B_{11}^{-1}\right)\sigma_1+\varepsilon_2^T(u)\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2 \, . \tag{1.31}$$

In (1.1), the symbol $T(u_i)$ represents the kinetic energy. In terms of the accepted notation, we write it as follows:

$$T(u_i)=\frac{1}{2}\iiint_V \rho\left(\dot{u}_r^2+\dot{u}_\theta^2+\dot{u}_z^2\right)dr d\theta dz \, , \tag{1.32}$$

where $\dot{u}_r$, $\dot{u}_\theta$, $\dot{u}_z$ – are velocities in the direction of the axes of the cylindrical coordinate system r, θ, z (Fig. 1), and ρ – is the density of the material from which the structure is made.

The resulting equations are equations of motion, since after varying (1.32) and integrating in parts over time, we obtain the variation of the kinetic energy of the anisotropic shell:

$$\delta T = \iint_{S_1} \rho\left[\dot{u}_r\delta\dot{u}_r + \dot{u}_\theta\delta\dot{u}_\theta + \dot{u}_z\delta\dot{u}_z\right]\Big|_{t_1}^{t_2} dS_1 - \int_{t_1}^{t_2}\iiint_V \rho\left(\ddot{u}_r\delta u_r+\ddot{u}_\theta\delta u_\theta+\ddot{u}_z\delta u_z\right)dV dt \ . \tag{1.33}$$

It is worth noting that in the work [21]the following expressions were used to approximate the displacements:

$$u(r,\theta,z,t)=\rho\cdot u(r,\theta,z)\cdot e^{-i\omega t} \tag{1.34}$$

and stresses:

$$\sigma_1^T=\sigma_1^T\cdot e^{-i\omega t}=\left[\sigma_r,\sigma_{r\theta},\sigma_{rz}\right]\cdot e^{-i\omega t} \ . \tag{1.35}$$

In (1.34) and (1.35) ω is the free vibration frequency of the anisotropic shell. Having performed the operation of differentiation (1.34) in time, we obtain the expressions for the accelerations:

$$\ddot{u}(r,\theta,z,t)=-\rho\omega^2\cdot u(r,\theta,z)\cdot e^{-i\omega t} \ . \tag{1.36}$$

The final form of the functional Π_1 presented in (1.1), is as follows:

$$\Pi_1 = \int_{t_1}^{t_2}\left\{\iiint_V \left[W(\sigma_1,\varepsilon)-T(u_i)\right]dV\right\}dt \, . \tag{1.37}$$

The expression for Π_1 is a part of the functional (1.1), since it replaces the number of independent parameters, due to the fulfillment of relations (1.28) and the condition $\varepsilon_2=\varepsilon_2(u)$. Then the variation of the functional (1.37), which is caused by a change in the components of the displacement eigenvector u and stresses σ_1 takes the form:

$$\delta\Pi_1=\int_{t_1}^{t_2}\left\{\iiint_V\left\{\left[-\frac{1}{2}\sigma_1^T B_{11}^{-1}\sigma_1+\left(\varepsilon_1^T(u)+\varepsilon_2(u)B_{11}^T B_{12}^{-1}\right)\sigma_1\right]\delta\sigma_1-\left[\frac{1}{2}\varepsilon_2^T\left(B_{22}-B_{12}^T B_{11}^{-1}B_{12}\right)\varepsilon_2\right]\delta\varepsilon_2+\right.\right.$$

$$+\left[\varepsilon_2^T(u)\left(B_{22}-B_{12}^T B_{11}^{-1} B_{12}\right)\varepsilon_2\right]\delta u - T(u)\delta u\Big\}dV\Big\}dt . \quad (1.38)$$

Then we will use the linear geometric relations that are given in [22]:

$$e_{rr}=\frac{\partial u_r}{\partial r}\,;\; e_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\,;\; e_{r\theta}=\frac{\partial u_\theta}{\partial r}-\frac{1}{r}u_\theta+\frac{1}{r}\frac{\partial u_r}{\partial \theta} . \quad (1.39)$$

In this case e_{rr} – are the relative linear deformations along the direction of the coordinate axis r, and e_{rz}, $e_{r\theta}$ – are relative shear strains tangent to the corresponding coordinate surfaces.

In accordance to the stationarity condition (1.38), when using the expressions for stresses $\sigma_1^T=(\sigma_{rr},\tau_{r\theta},\tau_{rz})$ displacements $u^T=(u_r,u_\theta,u_z)$, geometric relations (1.39), dependencies for the variation of kinetic energy (1.33), (1.34), (1.35), (1.36), as well as variations of external forces (1.38) and equating the expressions for independentvariations of stresses $\delta\sigma_{rr}$, $\delta\tau_{r\theta}$, $\delta\tau_{rz}$ and displacements δu_r, δu_θ, δu_z in the integral over the volume *V*, we obtain the following system of differential equations:

$$\frac{\partial\sigma_{rr}}{\partial r}=-\frac{c_{23}+1}{r}\sigma_{rr}-\frac{\partial\tau_{rz}}{\partial z}-\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{c_{22}}{r^2}u_r+\frac{c_{12}}{r}\frac{\partial u_z}{\partial z}+\frac{c_{26}}{r^2}\frac{\partial u_z}{\partial\theta}+\frac{c_{26}}{r}\frac{\partial u_\theta}{\partial z}+\frac{c_{22}}{r^2}\frac{\partial u_\theta}{\partial\theta}+\rho\omega^2 u_r\,;$$

$$\frac{\partial\tau_{rz}}{\partial r}=c_{13}\frac{\partial\sigma_{rr}}{\partial z}-\frac{1}{r}\tau_{rz}-\frac{c_{12}}{r}\frac{\partial u_r}{\partial z}-c_{11}\frac{\partial^2 u_z}{\partial z^2}-\frac{c_{66}}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}-\frac{c_{12}+c_{66}}{r}\frac{\partial^2 u_\theta}{\partial z\partial\theta}+$$

$$+\frac{c_{36}}{r}\frac{\partial\sigma_{rr}}{\partial\theta}-\frac{c_{26}}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2c_{16}}{r}\frac{\partial^2 u_z}{\partial z\partial\theta}-c_{16}\frac{\partial^2 u_\theta}{\partial z^2}-\frac{c_{26}}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\rho\omega^2 u_z\,;$$

$$\frac{\partial\tau_{r\theta}}{\partial r}=\frac{c_{23}}{r}\frac{\partial\sigma_{rr}}{\partial\theta}-\frac{2}{r}\tau_{r\theta}-\frac{c_{22}}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{c_{12}+c_{66}}{r}\frac{\partial^2 u_z}{\partial z\partial\theta}-c_{66}\frac{\partial^2 u_\theta}{\partial z^2}-\frac{c_{22}}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+$$

$$+c_{36}\frac{\partial\sigma_{rr}}{\partial z}-\frac{c_{26}}{r}\frac{\partial u_r}{\partial z}-c_{16}\frac{\partial^2 u_z}{\partial z^2}-\frac{c_{26}}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}-\frac{2c_{26}}{r}\frac{\partial^2 u_\theta}{\partial z\partial\theta}+\rho\,\omega^2 u_\theta\,;$$

$$\frac{\partial u_r}{\partial r}=c_{33}\sigma_{rr}+\frac{c_{23}}{r}u_r+c_{13}\frac{\partial u_z}{\partial z}+\frac{c_3}{r}\frac{\partial u_z}{\partial\theta}+c_{36}\frac{\partial u_\theta}{\partial z}+\frac{c_{23}}{r}\frac{\partial u_\theta}{\partial\theta}\,;$$

$$\frac{\partial u_z}{\partial r}=a_{55}\tau_{rz}+a_{45}\tau_{r\theta}-\frac{\partial u_r}{\partial z}\,;$$

$$\frac{\partial u_\theta}{\partial r}=a_{45}\tau_{rz}+a_{44}\tau_{r\theta}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{1}{r}u_\theta\,. \quad (1.40)$$

And r – is the radius of the cylinder, which does not depend on the coordinates z and θ (Fig. 1); σ_{rr}, τ_{rz}, $\tau_{r\theta}$ are the components of the stress tensor (1.7); u_z, u_θ, u_r – are the movements of the shell respectively along the axes z, θ, r; ω is the frequency of free vibration; ρ – is the density of the structure material. The constants c_{kl} (*k, l=1,* 2, 3, 6) are the characteristics of the shell material determined by the mechanical constants a_{kl} [14, 23]:

$$c_{11}=\frac{1}{|A_{22}|}\left(a_{22}a_{66}-a_{26}^2\right);\; c_{12}=\frac{1}{|A_{22}|}\left(a_{16}a_{26}-a_{12}a_{66}\right);\; c_{22}=\frac{1}{|A_{22}|}\left(a_{11}a_{66}-a_{16}^2\right);\; c_{16}=\frac{1}{|A_{22}|}\left(a_{12}a_{26}-a_{22}a_{16}\right);$$

$$c_{26}=\frac{1}{|A_{22}|}\left(a_{12}a_{16}-a_{11}a_{26}\right);\; c_{66}=\frac{1}{|A_{22}|}\left(a_{11}a_{22}-a_{12}^2\right);$$

$$|A_{22}|=a_{66}\left(a_{11}a_{22}-a_{12}^2\right)+a_{26}\left(a_{12}a_{16}-a_{11}a_{26}\right)+a_{16}\left(a_{12}a_{26}-a_{22}a_{16}\right)$$

$$c_{13}=a_{13}c_{11}+a_{23}c_{12}+a_{36}c_{16};\quad c_{23}=a_{13}c_{12}+a_{23}c_{22}+a_{36}c_{26};$$

$$c_{36}=a_{13}c_{16}+a_{23}c_{26}+a_{36}c_{66};\quad c_{33}=a_{33}-\left(a_{13}c_{13}+a_{23}c_{23}+a_{36}c_{36}\right). \quad (1.41)$$

Thus, using the variational equation (1.38), the three-dimensional system (1.40) of six homogeneous differential equations of motion of the linear theory of elasticity is derived. It is written in partial derivatives with respect to the six components of the amplitude values of the eigenvectors

$\sigma_1^T = (\sigma_{rr}, \tau_{r\theta}, \tau_{rz})$ and $u^T = (u_r, u_\theta, u_z)$ to determine the frequencies of free vibration of an anisotropic thick-walled composite cylindrical shell. To obtain it, we used the modified Ky-Washizu variational principle, which allows us to write down the boundary conditionsappropriatefor the equations. Using equations (1.40) and the appropriate boundary conditions, the frequencies of free vibrations in the three-dimensional formulation of a thick-walled composite anisotropic cylindricalshell can be determined. The derived system of homogeneous differential equations of motion (1.40) practically corresponds to the one given in Chapter 4 of the paper [2]. A certain difference lies in the use of Cauchy relations by the authors of this work according to [22].

In the case of problem to determine the frequencies of free vibrations, the solution of the system (1.40) must meet the conditions on the side surfaces when $r = r_1$:

$$\sigma_{rr}^0(r_1, z, \theta) = 0\,; \quad \tau_{rz}^0(r_1, z, \theta) = 0\,; \quad \tau_{r\theta}^0(r_1, z, \theta) = 0$$

and $r = r_2$

$$\sigma_{rr}^n(r_2, z, \theta) = 0\,; \quad \tau_{rz}^n(r_2, z, \theta) = 0\,; \quad \tau_{r\theta}^n(r_2, z, \theta) = 0\,. \qquad (1.42)$$

Conditions at the ends when $z = 0$, $z = L$ (Fig. 1) are assumed to be:

$$\sigma_{zz} = u_r = u_\theta = 0\,. \qquad (1.43)$$

Then we will ensure that the layers are in rigid contact for stresses and displacements:

$$\sigma_{rr}^i(r_i) = \sigma_{rr}^{i+1}(r_i); \quad \tau_{rz}^i(r_i) = \tau_{rz}^{i+1}(r_i); \quad \tau_{r\theta}^i(r_i) = \tau_{r\theta}^{i+1}(r_i);$$
$$u_r^i(r_i) = u_r^{i+1}(r_i); \quad u_z^i(r_i) = u_z^{i+1}(r_i); \quad u_\theta^i(r_i) = u_\theta^{i+1}(r_i). \qquad (1.44)$$

In this case, it is thenumber of the shell layer.

The conditions (1.43) correspond to the presence of a diaphragm at the edges of the cylinder that is absolutely rigid in its plane and flexible out of it from papers [2, 16].

2. Methodology for solving the problem

2.1. Reducing a three-dimensional system of equations of the theory of elasticity to a one-dimensional one

To solve the three-dimensional system of equations (1.40) under the appropriate conditions on the surfaces and ends (1.42), (1.43) of the cylindrical shell, we use the Bubnov–Galerkin method. According to it, we will expand all the functions into trigonometric series [16] along the cylinder's generative line *z* so that they satisfy the boundary conditions (1.43):

$$\sigma_{rr}(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{1,pk}(r)\cos k\theta + y'_{1,mk}(r)\sin k\theta\right]\sin l_m z\,;$$

$$\tau_{rz}(r,z,\theta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{2,pk}(r)\cos k\theta + y'_{2,mk}(r)\sin k\theta\right]\cos l_m z\,;$$

$$\tau_{r\theta}(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{3,pk}(r)\sin k\theta + y'_{3,mk}(r)\cos k\theta\right]\sin l_m z\,;$$

$$u_r(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{4,pk}(r)\cos k\theta + y'_{4,mk}(r)\sin k\theta\right]\sin l_m z\,;$$

$$u_z(r,z,\theta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[y_{5,pk}(r)\cos k\theta + y'_{5,mk}(r)\sin k\theta\right]\cos l_m z\,;$$

$$u_\theta(r,z,\theta) = \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\left[y_{6,pk}(r)\sin k\theta + y'_{6,mk}(r)\cos k\theta\right]\sin l_m z\,. \qquad (2.1)$$

In this wise, $y_{i,pk}$, $y'_{i,mk}$ (*i*=1–6) are the components, decomposed by trigonometric Fourier series, of the parts of the stress-strain state of the shell: σ_{rr}, τ_{rz}, $\tau_{r\theta}$, u_r, u_z, u_θ and *p*, *m*, *k* are the wave numbers in Fourier series, *k*is also the parameter of wave formation in the circumferential direction. The parameter $l_m = m\pi/L$, where L − is the length of the cylinder's generating surface (Fig. 1).

After mathematical transformations and separation of variables in equations (1.40) using relations (2.1), we obtain for each i-th layer an infinite system of ordinary differential equations in the normal Cauchy form:

$$\frac{d\overline{y}}{dr}=\left(T(r)-\omega^2 C(r)\right)\overline{y}\ ,\ T(r)=t_{n,l}(r)\ ,\ C(r)=c_{n,l}(r)\ ,\ n=\overline{1,\infty}\ ,\ l=\overline{1,\infty}\ , \tag{2.2}$$

where $\overline{y}=\left\{y_{1,p};y_{2,p};y_{3,p};y_{4,p};y_{5,p};y_{6,p};y'_{1,m};y'_{2,m};y'_{3,m};y'_{4,m};y'_{5,m};\ y'_{6,m}\right\}$ – is the solving eigenvector function, $T(r)$ – is a square matrix with variable coefficients that depends on the argument r, $C(r)$ – is a matrix characterizing the inertial properties of the shell, ω is the frequency of free vibration. Non-zero elements of the matrix $T(r)$ and the coefficients with unknowns of system (2.2) $t_{n,l}(r)$ have the following form.

$t_{1,1}=-\frac{c_{23}+1}{r}$, $t_{1,2}=l_p$, $t_{1,3}=-\frac{k}{r}$, $t_{1,4}=\frac{c_{22}}{r^2}$, $t_{1,5}=-l_p\frac{c_{12}}{r}$, $t_{1,6}=k\frac{c_{22}}{r^2}$, $t_{1,11}=\sum_{m=1}^{\infty}\varphi(p,m)k\frac{c_{26}}{r^2}$,

$t_{1,12}=\sum_{m=1}^{\infty}\varphi(p,m)\frac{c_{26}}{r}l_m$,

$t_{2,1}=c_{13}l_p$, $t_{2,2}=-\frac{1}{r}$, $t_{2,4}=-\frac{c_{12}}{r}l_p$, $t_{2,5}=c_{11}l_p^2+k^2\frac{c_{66}}{r^2}$, $t_{2,6}=-\frac{c_{12}+c_{66}}{r}kl_p$, $t_{2,7}=\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_{36}}{r}$,

$t_{2,10}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_{26}}{r^2}$, $t_{2,11}=\sum_{m=0}^{\infty}\varphi(m,p)2\frac{c_{16}}{r}kl_m$, $t_{2,12}=\sum_{m=0}^{\infty}\varphi(m,p)c_{16}l_m^2+\sum_{m=0}^{\infty}\varphi(m,p)c\frac{b_{26}}{r^2}k^2$,

$t_{3,1}=-\frac{c_{23}}{r}k$, $t_{3,3}=-\frac{2}{r}$, $t_{3,4}=\frac{c_{22}}{r^2}k$, $t_{3,5}=-\frac{c_{12}+c_{66}}{r}kl_p$, $t_{3,6}=c_{66}l_p^2+\frac{c_{22}}{r^2}k^2$, $t_{3,7}=\sum_{m=1}^{\infty}\varphi(p,m)c_{36}l_m$,

$t_{3,10}=-\sum_{m=1}^{\infty}\varphi(p,m)\frac{c_{26}}{r}l_m$, $t_{3,11}=\sum_{m=1}^{\infty}\varphi(p,m)\left(c_{16}l_m^2+\frac{c_{26}}{r^2}k^2\right)$, $t_{3,12}=\sum_{m=1}^{\infty}\varphi(p,m)2\frac{c_{26}}{r}kl_m$,

$t_{4,1}=c_{33}$, $t_{4,4}=\frac{c_{23}}{r}$, $t_{4,5}=-c_{13}l_p$, $t_{4,6}=k\frac{c_{23}}{r}$, $t_{4,11}=\sum_{m=1}^{\infty}\varphi(p,m)k\frac{c_{36}}{r}$, $t_{4,12}=\sum_{m=1}^{\infty}\varphi(p,m)c_{36}l_m$,

$t_{5,2}=a_{55}$, $t_{5,4}=-l_p$, $t_{5,9}=\sum_{m=0}^{\infty}\varphi(m,p)a_{45}$, $t_{6,3}=a_{44}$, $t_{6,4}=\frac{k}{r}$, $t_{6,6}=\frac{1}{r}$, $t_{6,8}=\sum_{m=1}^{\infty}\varphi(p,m)a_{45}$,

$t_{7,5}=-\sum_{m=1}^{\infty}\varphi(p,m)k\frac{c_{26}}{r^2}$, $t_{7,6}=\sum_{m=1}^{\infty}\varphi(p,m)\frac{c_{26}}{r}l_m$, $t_{7,7}=-\frac{c_{23}+1}{r}$, $t_{7,8}=l_p$, $t_{7,9}=\frac{k}{r}$, $t_{7,10}=\frac{c_{22}}{r^2}$,

$t_{7,11}=-l_p\frac{c_{12}}{r}$, $t_{7,12}=-k\frac{c_{22}}{r^2}$,

$t_{8,1}=-\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_{36}}{r}$, $t_{8,4}=\sum_{m=0}^{\infty}\varphi(m,p)k\frac{c_{26}}{r^2}$, $t_{8,5}=-\sum_{m=0}^{\infty}\varphi(m,p)2\frac{c_{16}}{r}kl_m$,

$t_{8,6}=\sum_{m=0}^{\infty}\varphi(m,p)c_{16}l_m^2+\sum_{m=0}^{\infty}\varphi(m,p)\frac{b_{26}}{r^2}k^2$, $t_{8,7}=c_{13}l_p$, $t_{8,8}=-\frac{1}{r}$, $t_{8,10}=-\frac{c_{12}}{r}l_p$, $t_{8,11}=c_{11}l_p^2+k^2\frac{c_{66}}{r^2}$,

$t_{8,12}=\frac{c_{12}+c_{66}}{r}kl_p$,

$t_{9,1}=\sum_{m=1}^{\infty}\varphi(p,m)c_{36}l_m$, $t_{9,4}=-\sum_{m=1}^{\infty}\varphi(p,m)\frac{c_{26}}{r}l_m$, $t_{9,5}=\sum_{m=1}^{\infty}\varphi(p,m)\left(c_{16}l_m^2+\frac{c_{26}}{r^2}k^2\right)$,

$t_{9,6}=-\sum_{m=1}^{\infty}\varphi(p,m)2\frac{c_{26}}{r}kl_m$, $t_{9,7}=\frac{c_{23}}{r}k$, $t_{9,9}=-\frac{2}{r}$, $t_{9,10}=-\frac{c_{22}}{r^2}k$, $t_{9,11}=\frac{c_{12}+c_{66}}{r}kl_p$, $t_{9,12}=c_{66}l_p^2+\frac{c_{22}}{r^2}k^2$,

$t_{10,5}=-\sum_{m=1}^{\infty}\varphi(p,m)k\frac{c_{36}}{r}$, $t_{10,6}=\sum_{m=1}^{\infty}\varphi(p,m)c_{36}l_m$, $t_{10,7}=c_{33}$, $t_{10,10}=\frac{c_{23}}{r}$, $t_{10,11}=-c_{13}l_p$, $t_{10,12}=-k\frac{c_{23}}{r}$,

$t_{11,3}=\sum_{m=0}^{\infty}\varphi(m,p)a_{45}$, $t_{11,8}=a_{55}$, $t_{11,10}=-l_p$,

$$t_{12,2}=\sum_{m=1}^{\infty}\varphi(p,m)a_{45},\ t_{12,9}=a_{44},\ t_{12,10}=-\frac{k}{r},\ t_{12,12}=\frac{1}{r}. \qquad (2.3)$$

Functions $\varphi(p,m)$ and $\varphi(m,p)$ depend on the integer parameters p and m and are defined by the following formulas:

$$\varphi(p,m)=\begin{cases}0, & \text{when } p+m \text{ – even number,}\\ \frac{2}{\pi}\left(\frac{1}{p-m}+\frac{1}{p+m}\right) & \text{when } p+m \text{ – odd number;}\end{cases}$$

$$\varphi(m,p)=\begin{cases}0, & \text{when } p+m \text{ – even number,}\\ \frac{2}{\pi}\left(\frac{1}{m-p}+\frac{1}{m+p}\right) & \text{when } p+m \text{ – odd number.}\end{cases}$$

The non-zero elements of the matrix $C(r)$ have the form:

$$c_{1,4}=\rho,\quad c_{2,5}=\rho,\quad c_{3,6}=\rho,\quad c_{7,10}=\rho,\quad c_{8,11}=\rho,\quad c_{9,12}=\rho. \qquad (2.4)$$

To the system of equations (2.2), it is necessary to attach the corresponding equations characterizing the conditions for fixing the shell surfaces perpendicular to the integration direction:

$$B_1\bar{y}(r,\omega^2)=0;\quad r=r_1; \qquad (2.5)$$

$$B_2\bar{y}(r,\omega^2)=0;\quad r=r_2, \qquad (2.6)$$

where B_1, B_2 – are rectangular matrices formed on the basis of the given boundary conditions on the shell surfaces (1.42).

The solution of the boundary value problem (2.2), (2.5) - (2.6) involves finding the natural frequencies of free vibrations ω and their corresponding forms $\bar{y}(r_i)$, i=1, 2 as harmonic components of the dynamic state. In the matrix T the decomposition of the sought eigenvectors into Fourier series in the circular coordinate is taken into account by the parameter k=0, 1, 2,... of the shell wave generation. As a result, the problem (2.2), (2.5), (2.6) according to [1, 2, 6, 10, 11, 24] leads to the solution of one-dimensional sequence of boundary value problems for eigenvalues of the following form:

$$\frac{d\bar{y}_k}{dr_1}=(T_k(r_1)-\omega_k^2C(r_1))\bar{y}_k, \qquad (2.7)$$

$$B_1\bar{y}_k(r,\omega^2)=0;\quad r=r_1; \qquad (2.8)$$

$$B_2\bar{y}_k(r,\omega^2)=0;\quad r=r_2, \qquad (2.9)$$

Thus, the determination of the eigenvalues of the problem (2.7 – 2.9) is carried out by selecting the values $\lambda=\omega^2$ at which it has a non-trivial solution $y\neq 0$. This corresponds to the fulfillment of the condition $det\,D=0$, where D – is the matrix of a homogeneous system of linear algebraic equations satisfying the boundary conditions (2.9). For each given value λ by the method of discrete orthogonalization [1, 2, 3, 16], the boundary value problem is solved and $det\,D$ is determined until its minimum value is found $|det\,D|$, which is closest to zero, in accordance with the specified accuracy of solving the problem.

This approach was also used by the authors in the papers [10, 11, 15], where it was accordingly adapted to solve the problems of calculating stress-strain states, stability, and determining the parameters of free vibrations of anisotropic thin and thick-walled composite cylindrical shells.

3. Results

The results of the reliability of determining the free vibration frequencies will be tested on the example of thin cylindrical shells made of isotropic and orthotropic materials [1] for:

1. A one-layer shell made of an isotropic material with the following geometric and mechanical characteristics: length along the generating axis L=2,0m; radius of the median surface R=1,0m; thickness h=0,01m; $E=E_0$;ν=0,3; $\rho=\rho_0$.

2. A one-layer shell made of orthotropic material with the following characteristics: length L=3,0m; radius of the middle surface R=1,0m; thickness h=0,01m; E_{zz}=176E_0; $E_{\theta\theta}$=176E_0; E_{rr}=7E_0; $G_{z\theta}$=3,5E_0; $G_{rz}=G_{r\theta}$=1,4E_0; $\nu_{z\theta}=\nu_{rz}=\nu_{r\theta}$=0,25; ρ =2 ρ_0 .

Table 1 shows the values for the considered shell structures $\bar{\omega}=\omega^2(\rho_0/E_0)\cdot 10^7$ corresponding to the natural vibration frequencies and wave formation parameters k in the circular direction obtained according to [1] and the proposed approach.

Table 1

Comparison of the results of the study of free vibrations of cylindrical shell

Shell variant	Result from [1]		Proposed approach		Convergence Δ, %
	$\bar{\omega}$	k	$\bar{\omega}$	k	
1	13,8	5	13,8	5	0,0
2	214	3	214	3	0,0

The analysis of the comparison of the results of natural vibrations and their corresponding forms, presented in Table 1, show the coincidence of the parameters under comparison.

We also compared the capabilities of the proposed approach with the results of the spatial calculation presented in the paper [6]. We considered a cylindrical shell with a length of L=0,6m, an inner surface radius r_1=0,26m of a layered structure with geometric and mechanical characteristics of the layers presented in Table 2. Moreover, layers 2 and 3 are reinforced rubber cords stacked in such a way that the main elastic directions of these layers are located at cross/angle-ply$\psi=\pm 65^0$ to the cylinder's base. The minimum frequency of natural vibrations of an anisotropic cylinder was determined and the case of axisymmetric natural vibrations when $k=0$ wasconsidered. A comparison of the results of the approach [6] with the proposed one is given in Table 3.

Table 2

Geometrical and mechanical characteristics of an anisotropic cylindrical shell

№ layer	$h\cdot 10^3$, m	E_{zz}, MPa	$E_{\theta\theta}$, MPa	$\nu_{z\theta}$	$G_{z\theta}$, MPa	$G_{rz}=G_{r\theta}$, MPa	$\rho\cdot 10^{-3}$, kg/m^3
1	0,75	5,81 10^2	1,18 10	0,42	4,33	3,01	1,00
2	1,40	1,84 10^3	7,36	0,47	2,08	1,88	1,54
3	1,40	1,84 10^3	7,36	0,47	2,08	1,88	1,54
4	5,86	5,00	5,00	0,49	1,68	1,68	1,00

Table 3

Results of comparison of frequencies of free vibrationsand parameters of wave formation

The number of waves in a circular direction, k	According to [6] $\omega/2\pi$	Proposed approach $\omega/2\pi$	Δ, %
0	5,208	5,203	0,1
4	0,693	0,681	1,7

Comparison of the results presented in Table 3, since their maximum discrepancy does not exceed 2%, indicates a satisfactory coincidence of the frequencies of natural vibrations obtained by the proposed approach compared to the data presented in the paper[6].

A comparative analysis of the obtained results of determining the minimum frequencies of free vibrations with the values of the quantities given in [25] was carried out. In this article, to determine the frequencies of free vibrations of orthotropic shell structures in the refined formulation, shear deformations along the thickness were taken into account.When comparing our results with [25], a cylindrical graphite/epoxy orthotropic shell with the following mechanical material properties was considered: E_{zz}=138 GPa; $E_{\theta\theta}$=8,9 GPa; E_{rr}=8,9 GPa; $G_{z\theta}$=G_{rz}=5,17 GPa; $G_{r\theta}$=2,89 GPa; $\nu_{z\theta}$=0,3; ρ=1600 kg/m^3. The geometric characteristics are as follows: the radius of the middle surface R=0,1905 m, the ratio of the length to the radius of the median surface L/R=1,0; ratio of radius to thickness R/h=25. Table 4 shows a comparison of the reduced values $\overline{\omega}$ of the minimum frequencies of vibrations and waves in the circular direction k of such a shell.

Table 4

Results of comparison of frequencies of free vibrationsand parameters of wave formation

The number of waves in a circular direction, k	According to [25] $\overline{\omega}$	Proposed approach $\overline{\omega}$	Δ, %
4	18,426	17,521	4,9

A comparison of the presented results indicates the coincidence of the results obtained by the proposed approach with those obtained according to the refined theory.

4. Discussion

As an implementation of the capabilities of the approach proposed in this paper, we consider the problem of determining the minimum frequencies of free vibrations for a cylindrical anisotropic shell generated from unsymmetric lay-ups. The minimum frequencies depending on the angle-ply ψ of the main directions of elasticity of a unidirectional fiber composite relative to the system of the structure's own axes. At the same time, we investigate the change in the frequencies of free vibrations with an increase in the number of layers, when orthotropy axes are located at cross/angle-ply $\pm\psi$ relative to the cylinder base. We compare the obtained frequencies with those obtained in the calculation of the same anisotropic shell according to the orthotropic scheme, when the mechanical constants (1.41) of the accepted generalized Hooke's law is $c_{16}=c_{26}=c_{36}=a_{45}=0$.

The geometric dimensions of the structure are as follows: length L=1,2 m, radius of the central surface r=0,6 m. Figures 2 and 3 present the results for the ratio of the thickness h to the radius of the middle surface h/R=1/5 and h/R=1/10, respectively. Mechanical characteristics of the material in its own axes of orthotropy: E_{zz}=280E_0, $E_{\theta\theta}$=E_{rr}=31E_0, $G_{z\theta}$=$G_{r\theta}$=10,5E$_0$, G_{rz}=21,2E_0, $\nu_{\theta z}$=0,25, $\nu_{z\theta}$=0,0277, E_0=1000 MPa, ρ=2118 kg/m^3.

The results of the studies of the reduced values of the free vibration frequencies $\overline{\omega}=\omega\cdot 1{,}81\cdot L\cdot 10^2\sqrt{\rho/E_0}$ of an anisotropic thick-walled cylindrical shell are shown in Fig. 2 and 3, where number 1 (solid line)indicates the graphs of the results obtained for an anisotropic one-layer cylinder, 2 (dashed line) − indicates a two-layer cylinder, 4 (dotted line)is for four-layer cylinder, and 8 (dash dotted line) - eight-layer cylinder. Curve 1$'$(solid line) characterizes the orthotropic approach to the calculation (OPR) of a thick-walled cylinder.

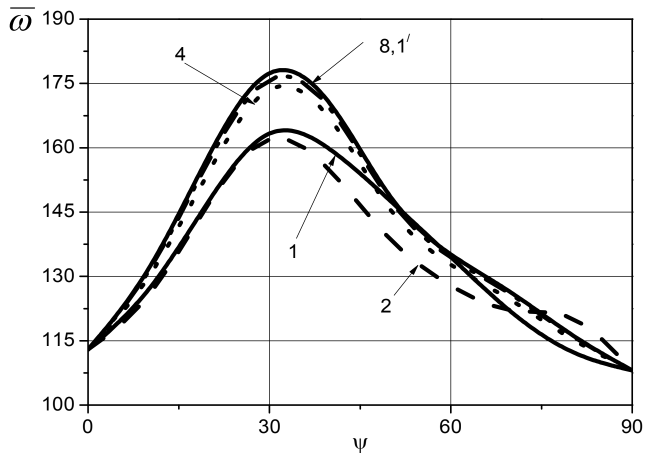

Fig. 2. Values of minimum frequencies of free vibrations of layered shells at h/r=1/5

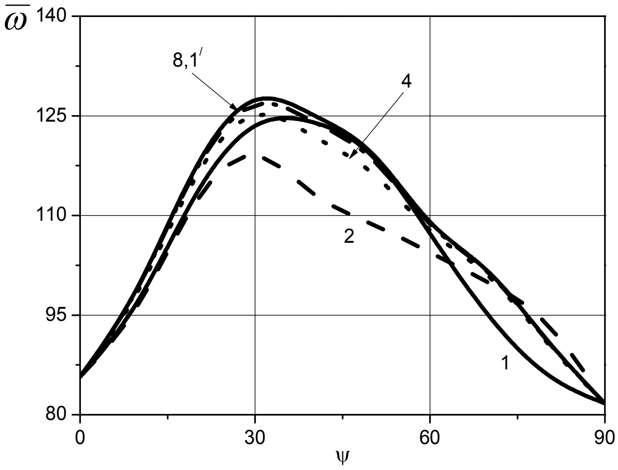

Fig. 3. Values of minimum frequencies of free vibrationsof layered shells at *h/r*=1/10

From the analysis of Fig. 2, 3 the following conclusions can be drawn; the minimum frequencies of free vibrations $\overline{\omega}$ depend on the change in the angle-ply ψ of the main directions of material elasticity. At the same time, regardless of the number of layers and thickness, the highest values of free vibration frequencies $\overline{\omega}$ for the considered anchoring conditions, occur for $20^0 \leq \psi \leq 40^0$ both for the case when considering the components of the generalised Hooke's law for this type of anisotropy and for the orthotropic calculation.

It should be mentioned that the values of the free vibration frequencies are solved using the orthotropic approach and do not depend on the number of layers, i.e., they are unchanged and higher than those found when taking into account all components of the generalized Hooke's law for the reduced type of anisotropy and all considered structural layered packages except for the two-layer one, where the opposite effect is observed at $\psi=80^0$ for the considered thicknesses. In this case, the maximum differences between the frequency values in the anisotropic calculation of a one-layer package compared to the orthotropic approach are 9,0%, and for a two-layer package they reach 11,6%. A further increase in the number of layers cross/angle-ply $\pm\psi$ (Fig. 1) of a thick-walled cylindrical shell to seven or eight leads to a decrease in the discrepancy to 0,9% between the anisotropic and orthotropic approaches to determining the minimum free vibration frequencies. This is, to some extent, a confirmation of the results presented in the papers [2, 15] for the calculations of thin anisotropic cylindrical shells for stability and free vibration, respectively. Thus, thick-walled anisotropic cylindrical shells can be calculated as orthotropic when the number of theircross/angle-ply layers increases to seven or more.

Figure 4 shows graphical dependences obtained for one-, two-, three- and four-layer anisotropic composite laminates generated from unsymmetric lay-ups, as well as orthotropic cylindrical shells constructed for the ratio of length to the radius of the middle surface *L/R*=2; 4; 10.The radius of the middle surface *r*=0,6m is unchanged, the ratio of the shell thickness h to its radius of the middle surface r is equal to 1/5. The conditions for fixing the ends correspond to (1.43).

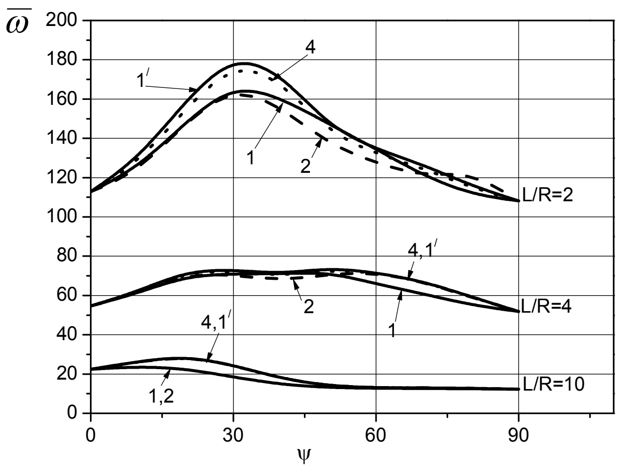

Fig. 4.Values of the minimum frequencies of free vibrations $\overline{\omega}$ for shells with a ratio of length to radius *L/R*=2; 4; 10

In Figure 4 the curves 1 (solid line) describe the values of minimum frequencies of free vibrations $\overline{\omega}$ for single-layer cylindrical shells generated from unsymmetric lay-ups of composite laminates.

In Figure 4 the curves 2 (dashed line) characterize the results of the study of two-layer structures, and the curves 4 (dashed line) - four-layer ones. These graphs are constructed according to the values obtained according to the proposed approach. The curves $1^/$ (solid line) characterize the results of determining the minimum frequencies for such shells according to the orthotropic approach (OPR). At the Figure 4 are presented the results of calculations for shells with ratios of length to the radius of the middle surface *L/R*=2, 4, 10.

When analyzing the results shown in Figure 4 it should be noted, that relatively short anisotropic cylinders with a ratio of length to radius *L/R*=2 for each individual case of the structural structure of the shell wall (one-, two-, four-layer) along the entire numerical axis ψ have the highest values of the minimum frequencies of free vibrations compared to the ratios *L/R*=4 and *L/R*=10. In the case of shells with *L/R*=4, the minimum frequency values are between the same as those obtained for cylinders with *L/R*=2 and *L/R*=10 ratios.

5. Conclusions

In this paper, a three-dimensional system of homogeneous partial differential equations of motion of the linear theory of elasticity of an anisotropic body in the cylindrical coordinate system was obtained, based on the modified Ky-Washizu variational principle. To reduce it to a one-dimensional principle, double trigonometric series were used, where the unknowns along the generating and circumferential directions of the thick-walled shell were approximated using the Bubnov – Galerkin analytical method. The numerical method of discrete orthogonalization was applied to solve the resulting one-dimensional problem in the direction perpendicular to the middle surface of the shell structure.

The dependence of the minimum frequencies of free vibrations of an anisotropic cylindrical shell on the number of layers with the main elasticity directions located at cross/angle-ply $\pm\psi$ to the parent one, from the thickness and length of the shell is investigated. It is proved that when the number of layers exceeds seven, the frequencies of free vibration of a thick-walled orthotropic and anisotropic cylindrical shell are practically the same.

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Трач В.М., Подворний А.В., Жукова Н.Б., Бондарський О.Г.
ВІЛЬНІ КОЛИВАННЯ ШАРУВАТИХ АНІЗОТРОПНИХ ТОВСТОСТІННИХ ЦИЛІНДРИЧНИХ ОБОЛОНОК

В статті представлено підхід щодо розрахунку на вільні коливання товстостінних пружних шаруватих анізотропних циліндричних оболонок. Анізотропія обумовлена використанням матеріалу, пружні характеристики якого знаходяться в одній площині, що паралельна серединній поверхні оболонки. Такий вид анізотропії виникає з-за не співпадіння головних напрямів пружності ортотропного волокнистого композита з осями циліндричної системи координат.

Розрахунки, що описують вільні коливання товстостінних циліндричних анізотропних оболонок, реалізовані шляхом виведення системи з шести диференціальних рівнянь руху в частинних похідних просторової лінійної теорії пружності. Для цього авторами, відповідним чином, модифіковано варіаційний принцип Ху-Васідзу, що дозволяє записувати не лише рівняння руху, а й відповідні їм граничні умови. При використанні аналітичного методу Бубнова – Гальоркіна, отримано нескінчену одновимірну систему диференціальних рівнянь нормального виду Коші, що дозволяє знаходити частоти вільних коливань товстостінних шаруватих анізотропних циліндричних оболонок. Невідомі в системі рівнянь руху, що описують параметри напружено-деформованого стану оболонок, вибрані за радіальним напрямком. Для реалізації одновимірної розв'язуючої системи диференціальних рівнянь циліндричних оболонок використано числовий метод дискретної ортогоналізації, який авторами було відповідним чином адаптовано. На цій основі написано алгоритм і створено програмний комплекс для персональних комп'ютерів, що дозволяє розв'язувати проблемистосовно встановлення параметрів вільних коливань товстостінних шаруватих анізотропних композитних циліндричних оболонок.

Представлено розв'язки задач про вплив на частоти вільних коливань анізотропної товстостінної циліндричної оболонки при урахуванні: кута повороту головних напрямів пружності ортотропного волокнистого композита; збільшення кількості перехресно-укладених шарів; зміни геометричних параметрів конструкції; чотирьох видів граничних умов.

Ключові слова: циліндричні анізотропні оболонки, вільні коливання, тривимірна система рівнянь руху, варіаційний принцип Ху-Васідзу, метод Бубнова-Гальоркіна.

Trach V.M., Podvornyi A.V., Zhukova N.B., Bondarskyi O.G.
FREE VIBRATIONS OF LAYERED ANISOTROPIC THICK-WALLED CYLINDRICAL SHELLS

The article presents an approach to the calculation of free oscillations of thick-walled elastic layered anisotropic cylindrical shells. Anisotropy is due to the use of material whose elastic characteristics are in one plane parallel to the middle surface of the shell. This type of anisotropy arises due to the non-coincidence of the main directions of elasticity of the orthotropic fibrous composite with the axes of the cylindrical coordinate system.

Calculations describing the free oscillations of thick-walled cylindrical anisotropic shells are implemented by deriving a system of six differential equations of motion in partial derivatives of the spatial linear theory of elasticity. For this purpose, the authors modified the Hu-Washizu variational principle accordingly, which allows writing not only the equations of motion, but also the boundary conditions corresponding to them. When using the Bubnov-Galyorkin analytical method, an infinite one-dimensional system of differential equations of the normal Cauchy form is obtained, which allows finding the frequencies of free oscillations of thick-walled layered anisotropic cylindrical shells. The unknowns in the system of equations of motion describing the parameters of the stress-strain state of the shells are selected in the radial direction. To implement a one-dimensional solving system of differential equations of cylindrical shells, the numerical method of discrete orthogonalization was used, which was adapted accordingly by the authors. On this basis, an algorithm was written and a software complex was created for personal computers, which allows solving problems related to setting parameters of free oscillations of thick-walled layered anisotropic composite cylindrical shells.

The solutions of problems on the influence on the frequencies of free oscillations of an anisotropic thick-walled cylindrical shell are presented, taking into account: the angle of rotation of the main directions of elasticity of an orthotropic fibrous composite; increasing the number of cross-stacked layers; changes in the geometric parameters of the structure; four types of boundary conditions.

Key words: cylindrical anisotropic shells, free vibrations, three-dimensional system of equations of motion, Ky-Washizu variational principle, Bubnov–Galerkin method.

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Трач В.М., Подворний А.В., Жукова Н.Б., Бондарський О.Г. **Вільні коливання шаруватих анізотропних товстостінних циліндричних оболонок** // Опір матеріалів і теорія споруд: наук.-тех. збірн. – К.: КНУБА, 2024. – Вип. 113 – С. 250 – 264.
Запропоновано підхід до розв'язку задачі про вільні коливання шаруватих товстостінних циліндричних оболонок з анізотропного матеріалу, пружні характеристики якого знаходяться в одній площині, що є дотичною серединній поверхні. Спираючись на модифікований варіаційний принцип Ху-Васідзу, виведена тривимірна система однорідних диференціальних рівнянь руху в частинних похідних лінійної теорії пружності анізотропного тіла в циліндричній системі координат і відповідні їй граничні умови. Для пониження розмірності тривимірної системи застосовано аналітичний метод Бубнова-Гальоркіна. Це дозволяє визначати частоти вільних коливань товстостінних шаруватих анізотропних циліндричних оболонкових конструкцій.На основі розробленого підходу до розрахунку вільних коливань товстостінних анізотропних циліндричних оболонок у просторовій постановці проведено аналіз результатів визначення частот вільних коливань.Запропонований підхід суттєво розширює можливості розрахунку оболонкових конструкцій із композиційних матеріалів.
Табл. 4. Іл. 4. Бібліогр. 25 назв.

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Trach V.M., Podvornyi A.V., Zhukova N.B., Bondarskyi O.G. **Free vibrations of layered anisotropic thick-walled cylindrical shells** // Strength of Materials and Theory of Structures: Scientific-and-technical collected articles. – Kyiv: KNUBA, 2024. – Issue 113. – P. 250-264.
In this work an approach is proposed to solve the problem of free vibrations of layered thick-walled cylindrical shells made of an anisotropic material, the elastic characteristics of which are in the same plane, tangent to the middle surface. A three-dimensional system of homogeneous differential equations of motion of the linear theory of elasticity of an anisotropic body on the basis of the modified Ky-Washizu variational principle was developed. It was recorded in the cylindrical coordinate system for the appropriate boundary conditions on the surfaces and ends of the shell. Using the analytical Bubnov – Galerkin method to reduce the dimension of a three-dimensional system, an approach to obtaining an infinite one-dimensional system of differential equations is presented. It gives possible to determine the frequencies of free vibrations of thick-walled unsymmetric laminate anisotropic cylindrical shell structures. Based on the developed approach to the calculation of free vibrations in the spatial setting of a thick-walled anisotropic cylindrical shell, an analysis of the results of frequency determination was carried out. The proposed approach significantly expands the possibilities of calculating shell structures from composite materials.
Tabl. 4. Fig. 4. Ref. 25

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