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NUMERICAL-ANALYTICAL APPROACH TO SOLVING PROBLEMS OF NON-STATIONARY THERMAL CONDUCTIVITY OF A NON-THIN ANNULAR PLATE

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This paper considers the first stage of calculating the initial boundary value problem of non-stationary thermal conductivity of cylindrical bodies using a modified method of lines, namely dimension reduction of the original differential equations, initial and boundary conditions. The original equations of thermal conductivity are defined in a cylindrical coordinate system in a spatial setting. An object is a cylindrical body with commensurate dimensions. This area of research is relevant, because when calculating the load bearing elements of structures to thermal effects, the first step is to determine the distribution of temperature fields. Boundary conditions are considered as conditions of convective heat transfer, which by means of boundary transition are transformed into boundary conditions of the first and second types.

Keywords: thermal conductivity, convective heat transfer, dimensionality reduction, modified method of lines, projection method, reduced equations, trigonometric series, basic functions.

One of the important computational models of rotating bodies under the action of forceful, kinematic and thermal influences is a non-thin annular plate. Before calculating the annular plate for thermal effects, the distribution of the thermal field over the volume of the plate is determined (calculation of the thermal conductivity problem). The calculated functions are considered in the cylindrical coordinate system and depend on three spatial variables and a time variable (Fig. 1).

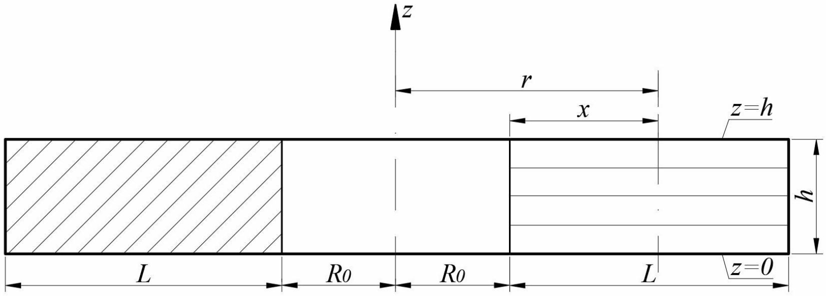

Fig. 1 Annular plate

The original equations of thermal conductivity in the cylindrical coordinate system are considered as a system of differential equations of the first order:

Fourier's Law:

$$q_r = -\lambda_T \cdot \frac{\partial T}{\partial r}; \quad q_\theta = -\lambda_T \cdot \frac{1}{r} \cdot \frac{\partial T}{\partial \theta}; \quad q_z = -\lambda_T \cdot \frac{\partial T}{\partial z}. \tag{1}$$

Heat balance equation:

$$\rho c \cdot \frac{\partial T}{\partial t} = -\frac{q_r}{r} - \frac{\partial q_r}{\partial r} - \frac{1}{r} \cdot \frac{\partial q_\theta}{\partial \theta} - \frac{\partial q_z}{\partial z} + Q_T \tag{2}$$

or in the form of a second-order differential equation in spatial coordinates:

$$\frac{\rho c}{\lambda_T}\cdot\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial T}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}. \tag{3}$$

In connection with the further application of the method of lines, the variable r is replaced by x, taking into account the ratio:

$$r=R_0+x\,. \tag{4}$$

Due to the fact that (4) is a linear relation, the derivatives are equal:

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\,. \tag{5}$$

Therefore, equations (1), (2) are transformed into:

$$q_r=-\lambda_T\cdot\frac{\partial T}{\partial x};\;\; q_\theta=-\lambda_T\cdot\frac{1}{R_0+x}\cdot\frac{\partial T}{\partial \theta};\;\; q_z=-\lambda_T\cdot\frac{\partial T}{\partial z}-\text{ Fourier's Law,} \tag{6}$$

$$\rho c\cdot\frac{\partial T}{\partial t}=-\frac{q_r}{R_0+x}-\frac{\partial q_r}{\partial x}-\frac{1}{R_0+x}\cdot\frac{\partial q_\theta}{\partial \theta}-\frac{\partial q_z}{\partial z}+Q_T\text{ - heat balance equation.} \tag{7}$$

The used replacement of $r\to x$ variables allows to expand the capabilities of the algorithm. Equations (6), (7) for $R_0\to\infty$ are transformed into the equations of the plane problem of thermal conductivity in the Cartesian coordinate system (x,z).

The Modern numerical-analytical methods are used to construct approximate solutions of multidimensional problems using dimensionality reduction of the original calculation equations.

To set the initial problem, it is necessary to add the boundary and initial conditions to the original equations, which must satisfy the calculated functions.

On the boundary surfaces of the object, the calculation functions $T(x,\theta,z,t)$ on spatial coordinates (on the $z=0$, $z=h$, and $x=0$, $x=L$ surfaces) are considered as conditions of convective heat exchange with the surrounding environment. It is considered that $T(x,\theta,z,t)$ depends on variables as on parameters.

When $x=0$:

$$q_r(0,\theta,z,t)=q^0_{r,c}(z,t)-\alpha^0[T(\theta,z,t)-\theta^0_r(\theta,z,t)], \tag{8}$$

When $x=L$:

$$q_r(L,\theta,z,t)=q^L_r(\theta,z,t)-\alpha^L[T(L,\theta,z,t)-\theta^L_c(\theta,z,t)]; \tag{9}$$

When $z=0$:

$$q^0_z(x,\theta,t)=-\alpha^-[T(x,\theta,h^-,t)-\theta^-_c(x,\theta,t)], \tag{10}$$

When $z=h$:

$$q_{z,c}(x,\theta,h^+,t)=q_z(x,\theta,h,t)+\alpha^+[T(x,\theta,h,t)-\theta^+_c(x,\theta,t)]\,. \tag{11}$$

With respect to variable θ, the calculated functions must satisfy the periodicity conditions, that is:

$$T(\theta+2\pi)=T(\theta)\,. \tag{12}$$

On the time variable, the temperature function must satisfy the initial condition, that is:

$$T(x,\theta,z,0)=T_0(x,\theta,z), \tag{13}$$

where $T_0(x,\theta,z)$ is the known temperature distribution function over all points of the body volume.

When the dimensionality of the original equations is reduced by two spatial variables (by z and θ.), the reduced equations, boundary and initial conditions define a one-dimensional function. This function also depends on the time coordinate. For this type of problem, numerical methods of mathematical physics have developed effective numerical algorithms that are easily adapted to reduced problems.

To reduce dimensionality, a modified method of lines, developed by the authors of this paper [5], [6], is used.

Parallel straight lines with a constant step are applied to the area of the body being examined.

In this case, these are lines parallel to the $0y$ - axis.

Each such line corresponds to piecewise linear locally concentrated basis functions [1], which depend on the z - coordinate. Any calculated function is a linear combination of basis functions and depends on other variables as parameters:

$$T(x,\theta,z,t) = T^i(x,\theta,t)\cdot\varphi_i(z)\,. \tag{14}$$

For repeating indices, summation is assumed within the index values (Einstein matching).

The system of basis functions $\varphi_i(z)$ forms a linear space, the basis of which it is. On the elements of this linear space we determine the scalar product:

$$(f(z),g(z)) = \int_0^h f(z),g(z)dz\,. \tag{15}$$

The system of functions (Fig. 2) allows you to calculate the scalar products of any pair of basis functions according to "Vereshchagin's Rule".

The system of basis functions is "almost orthogonal".

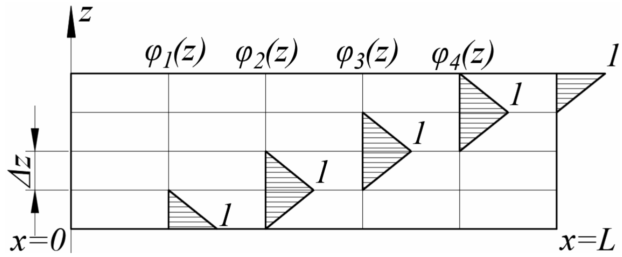

Fig. 2

Tensor calculus, which extends to multidimensional Euclidean spaces, has been developed for calculations of vector operations with oblique bases [2].

For further use of tensor operations, a "reciprocal basis" is used next to the selected "main basis" $\varphi_i(z),\ i=\overline{1,N_z}$.

The "reciprocal basis" is built on the main basis $[1_\delta]$ using the components of the metric tensor:

$$\varphi^i(z) = g^{ij}\cdot\varphi_j(z), \tag{16}$$

where g^{ij} is the twice contravariant metric tensor $g^{ij}=(\varphi^i(z),\varphi^j(z))$, calculated from the matrix components of the doubly covariant tensor $g_{ij}=(\varphi_i(z),\varphi_j(z))$ by the formula:

$$\{g^{ij}\} = \{g_{ij}\}^{-1}. \tag{17}$$

A function that depends on a variable z , can be written as a expansion on a main basis:

$$f(z) = f\cdot\varphi_i(z) \tag{18}$$

or in the form of expansion on a reciprocal basis:

$$f(z) = f_i\cdot\varphi^i(z). \tag{19}$$

The coefficients f^i and f_i in expansions (19) and (20) found by scalar multiplication by elements of the basic or reciprocal basis:

$$f^i = (f(z),\varphi^i(z));\ \ f_i = (f(z),\varphi_i(z)). \tag{20}$$

f^i - coefficients in the distribution according to the main base, f_i - moments relative to the main base.

In problems of the mechanics of axisymmetric bodies in a cylindrical coordinate system, the symmetry of influences is taken into account. This leads to a significant simplification of the problems.

With axisymmetric external influences, the thermal and mechanical states of the annular plate are axisymmetric. This simplifies the problem and calculation equations. The reduced equations are reduced to one-dimensional coordinates for which explicit or implicit difference schemes are used. If the temperature function is not axisymmetric, the problem becomes more complicated.

The Bubnov-Galyorkin-Petrov projection method with the basis functions selected above is used to reduce the dimensionality of the original equations using a modified method of lines [1].

Natural boundary conditions (8) - (11) are used as boundary conditions for variable z, then the basic functions may not satisfy the boundary conditions.

To obtain the reduced equations in the coefficients of z (corresponding coefficients in moments relative to z), all equations are multiplied scalar by $\varphi^i(z)$.

Calculated functions depend on four variables - $x,\theta,z,t,$ and the scalar product is related to integration over z, other variables of x,θ,t do not depend on z and are considered parameters, so the functions of these variables and their derivatives are carried out for the signs of integrals over z. Accordingly, we obtain:

$$((q_x=-\lambda_T\frac{\partial T}{\partial x}),\varphi^i(z))\to q_z^i(x,\theta,t)=-\lambda_T\frac{\partial T^i(x,\theta,t)}{\partial x}, \tag{21}$$

$$((q_\theta=-\lambda_T\frac{1}{R_0+x}\cdot\frac{\partial T}{\partial\theta}),\varphi^i(z))\to q_\theta^i(x,\theta,t)=\frac{-\lambda_T}{R_0+x}\cdot\frac{\partial T^i(x,\theta,t)}{\partial\theta}, \tag{22}$$

$$((q_z=-\lambda_T\frac{\partial T}{\partial z}),\varphi^i(z))\to q_z^i(x,\theta,t)=-\lambda_T\int_0^h\frac{\partial T}{\partial z}\cdot g^{ij}\cdot\varphi_j(z)dz=$$

$$=-\lambda_T\cdot g^{ij}\cdot\int_0^h\frac{\partial}{\partial z}(T^\alpha(x,\theta,t)\cdot\varphi_\alpha(z))\cdot\varphi_j(z)=-\lambda_T\cdot T^\alpha\cdot g^{ij}\cdot\int_0^h\varphi_j(z)\cdot\varphi_\alpha^{/}(z)dz=-\lambda_T\cdot g^{ij}\cdot b_{j\alpha}T^\alpha.$$

Here, the index-dropping operation is used $\varphi^i(z)=g^{ij}\cdot\varphi_j(z)$, transformation function $T(x,z,\theta,t)=T^\alpha(x,\theta,t)\cdot\varphi_\alpha(z);$ and the supporting matrix $\int_0^h\varphi_j(z)\cdot\varphi_\alpha^{/}(z)dz=b_{j\alpha}$.

The final result is:

$$q_z^i(x,\theta,t)=-\lambda_T\cdot g^{ij}\cdot b_{j\alpha}T^\alpha. \tag{23}$$

The main problem of applying the projection method to the solution of operator equations is the convergence and stability of the method during numerical implementation.

A large number of works are devoted to the question of the convergence of the projection method. Fundamental results were obtained by N.I. Polsky, S.G. Mikhlin, M.M Vinnik, M.A Krasnosel'skii, G.M. Vainikko, P.P. Zabreiko, Y.B. Rutitsky and others. A necessary condition for convergence is that the system of basis functions belongs to the energy space of the operator and its completeness in space.

Since in our approach the boundary conditions on the lateral surfaces are natural, the energy space of the operators consists of functions that may not satisfy the boundary conditions.

The general convergence theorems of the projection method in practical calculations establish the fact of convergence, but the assessment of the speed of convergence is more important. Construction of such estimates is not possible, therefore, the fact of convergence and its speed are investigated in the paper based on numerical experiments. This approach corresponds to the level of mathematical research adopted in structural mechanics.

As numerical experiments show, the reduction of dimensionality increases the level of rigidity of the reduced equations, which in some cases leads to the loss of stability of numerical methods.

But nowadays, numerical methods have been built that ensure the stability of numerical algorithms in solving Cauchy problems, boundary problems, etc. This must be taken into account when adapting existing numerical methods to the solution of reduced equations.

$$((\rho c\cdot\frac{\partial T}{\partial t}=-\frac{q_r}{R_0+x}-\frac{\partial q_r}{\partial x}-\frac{1}{R_0+x}\cdot\frac{\partial q_\theta}{\partial\theta}-\frac{\partial q_z}{\partial z}+Q_T),\varphi^i(z))\to$$

$$\to\rho c\cdot\frac{\partial T^i(x,\theta,t)}{\partial t}=\frac{q_r^i(x,\theta,t)}{R_0+x}-\frac{\partial q_r^i(x,\theta,t)}{\partial x}-\frac{1}{R_0+x}\cdot\frac{\partial q_\theta^i(x,\theta,t)}{\partial\theta}-((\frac{\partial q_z}{\partial z},\varphi^i(z))+Q_T^i(x,\theta,t). \tag{24}$$

The penultimate component is considered separately:

$$((\frac{\partial q_z}{\partial z},\varphi^i(z))=\int_0^h \frac{\partial q_z}{\partial z}\cdot\varphi^i(z)dz=\int_0^h \frac{\partial q_z}{\partial z}\cdot g^{ij}\cdot\varphi_j dz=g^{ij}\int_0^h \frac{\partial q_z}{\partial z}\cdot\varphi_j dz\,. \tag{25}$$

Since the function q_z is differentiated once, it is not possible to make a substitution of the schedule by base functions for it.

It is preliminarily recommended to "soften" the integration by means of integration by parts [1]. The necessary transformations look like:

$$\int_0^h \frac{\partial q_z}{\partial z}\cdot g^{ij}\cdot\varphi_j(z)dz=g^{ij}\cdot\int_0^h \frac{\partial q_z}{\partial z}\cdot\varphi_j(z)dz=g^{ij}\cdot\begin{vmatrix} u=\varphi_j(z);\ du=\varphi_j'(z)dz \\ dv=\dfrac{\partial q_z}{\partial z}dz;\ v=q_z \end{vmatrix}=$$

$$=(q_z\cdot\varphi_j(z))\Big|_0^h-\int_0^h q_z\cdot\varphi_j'(z)dz=g^{ij}(q_z^{\ N_z}\cdot\delta_j^{\ N_z}-q_z^{\ 1_z}\cdot\delta_j^{\ 1_z})-g^{ij}\int_0^h q_z^{\alpha}\cdot\varphi_\alpha(z)\cdot\varphi_j'(z)dz=$$

$$=g^{iN_z}\cdot q_z^{\ N_z}-g^{i1_z}\cdot q_z^{\ 1_z}-g^{ij}\cdot q_z^{\alpha}\int_0^h \varphi_\alpha(z)\cdot\varphi_j'(z)dz=g^{iN_z}\cdot q_z^{\ N_z}-g^{i1_z}\cdot q_z^{\ 1_z}-g^{ij}\cdot b_{\alpha j}\cdot q_z^{\alpha}\,.$$

Here is marked $b_{\alpha j}=\int_0^h \varphi_\alpha(z)\varphi_j^{/}(z)dz$. The final result is:

$$\int_0^h \frac{\partial q_z}{\partial z}\cdot g^{ij}\cdot\varphi_j(z)dz=g^{iN_z}\cdot q_z^{\ N_z}-g^{i1_z}\cdot q_z^{\ 1_z}-g^{ij}\cdot b_{\alpha j}\cdot q_z^{\alpha}\,. \tag{26}$$

The reduced equation of thermal equilibrium with respect to z has the form:

$$\rho c\frac{\partial T^i(x,\theta,t)}{\partial t}=-\frac{q_r^i(x,\theta,t)}{R_0+x}-\frac{\partial q_r^i(x,\theta,t)}{\partial x}-\frac{1}{R_0+x}\cdot\frac{q_\theta^i(x,\theta,t)}{\partial\theta}+$$

$$+g^{ij}\cdot b_{\alpha j}\cdot q_z^{\alpha}-g^{iN_z}\cdot q_z^{\ N_z}+g^{i1_z}\cdot q_z^{\ 1_z}+Q_T^i\,. \tag{27}$$

Excluding the components of the heat flow vector $q_r^i(x,\theta,z)$ and $q_z^i(x,\theta,z)$ from the reduced thermal equilibrium equations (27) with the help of the Fourier law equations (21), (22), (23), we obtain the equations relative to the temperature function:

$$\rho c\frac{\partial T^i(x,t)}{\partial t}=\frac{1}{R_0+x}\cdot\frac{\partial T^i(x,\theta,t)}{\partial x}+\frac{\partial^2 T^i(x,\theta,t)}{\partial x^2}+\frac{\partial^2 T^i(x,\theta,t)}{\partial\theta^2}+$$

$$+\tilde{Q}_T^i(x,\theta,t)-g^{ij}\cdot b_{\alpha j}\cdot g^{\alpha\beta}\cdot b_{j\alpha}\cdot T^{\alpha}, \tag{28}$$

where $\tilde{Q}_T^i(x,\theta,t)=Q_T^i(x,\theta,t)-g^{iN_z}\cdot q_z^{iN_z}+g^{i1_z}\cdot q_z^{i1_z}$.

Of particular interest in the problems of thermoelasticity of axisymmetric bodies are non-axisymmetric thermal influences. Under such influences, the thermal field and the stress-strain state are non-axisymmetric, that is, more difficult to research. In structural mechanics, for objects that have some variant of symmetry, a method of taking into account symmetry has been developed - an arbitrary load is considered as the sum of symmetrical and inversely symmetrical loading.

For an axisymmetric object with asymmetric load, the load is considered as the sum of three options: axisymmetric component (relative to the axis $0z$); locally symmetric component and locally skew-symmetric component. These components are described by mathematical methods. Thus, the axisymmetric component for the heat conduction equations is described by equations (27), which do not depend on the variable θ:

$$\frac{\partial T^i(x,t)}{\partial t}=\frac{1}{R_0+x}\cdot\frac{\partial T^i(x,t)}{\partial x}+\frac{\partial^2 T^i(x,t)}{\partial x^2}-\lambda_T\cdot g^{ij}\cdot b_{j\beta}\cdot T^{\beta}(x,t)\,. \tag{29}$$

The locally symmetric component of the load is described by a segment of the Fourier series with respect to variable θ (cosine series expansions). Local skew-symmetric component - sine series expansions. In all the cases mentioned above, in this work, a method is constructed to simplify the initial problem by dimension reduction with respect to one more spatial coordinate - θ. An option is where the external thermal influences are not axisymmetric and can then be written as a Fourier series with respect to the vertical axis, which includes the axisymmetric part of the influence, the cosine expansion, and the sine expansion. Three problems are considered - axisymmetric problem, calculation on cosine influence and calculation on sinusoidal influence. The computational equation in the last two cases includes an operator $\frac{\partial^2}{\partial\theta^2}$, that acts on functions that satisfy the boundary conditions of periodicity. It is not very convenient to apply the Fourier series method here, because it is necessary to reduce the computational equations to homogeneous equations. Therefore, it is convenient to reduce the dimensionality with respect to another spatial variable - θ by means of a finite integral transform [3], [4], using the eigenfunctions of the differential operator of the boundary value problem with respect to variable θ. The role of integral transform in solving problems of mathematical physics is very large. They are used as the main tool for obtaining analytical solutions of evolutionary and stationary equations defined in bounded and unbounded domain. Thus, in all cases, the original equation is scalarly multiplied by the transform kernel, and then integrated over the domain of this equation. To transform the integral relations obtained, integration by parts is used, while components outside the integral sign are excluded using the initial and boundary conditions. Each of the methods using integral transform differs from the others in the domain of changing the functions on which the transformation is defined and in the kind of kernel. The operational method, based on the application of the Laplace transform [1], is chronologically the first of all the methods considered. It is used to convert differential equations by time to algebraic relations and therefore applies to equations defined in a semi-infinite interval. The main achievements of the operational method - a large table of correspondences between the originals and mappings. After solved, it's easy to go back to the originals. It should be noted that the operating method is successfully used in the spatial coordinates defined on a semi-infinite interval. However, in spatial coordinates, the Fourier transform (on a semi-infinite interval) is often used to transform equations. For the transformation of differential equations defined on finite domains, the so-called finite integral transforms are used [4], [3].

Formally, the finite integral transform method is similar to the projection method. But from the point of view of modern mathematics there is a significant difference between them, which consists in the application in the finite integral transform eigenfunction method of the operator of the boundary value problem under consideration. In this regard, the Finite integral transform method uses the appropriate terminology, for example - "original" and "image", which is accepted in all integral transforms for equations written in a cylindrical coordinate system and many others. To transform equations defined on finite domains, finite integral transforms are applied. The finite integral transform method [1] appeared for two reasons. First, it was necessary to give a standard form to the transforms that accompany the solution of differential equations in spatial coordinates by the eigenfunction method. The specific properties of the eigenfunctions of self-adjoint operators of mathematical physics, especially their orthogonality, allow us to extend the ideas of the operational method to boundary value problems. The second reason is the desire to free the use of eigenfunctions from the complications associated with the heterogeneity of boundary conditions. In the eigenfunction method, the boundary conditions must be reduced to homogeneous ones. For this purpose, an auxiliary particular solution is found that satisfies the initial boundary conditions. It's not always easy to do. The finite integral transform method does not require this complication. Here, the inhomogeneity in the boundary conditions is taken into account in the components outside the integral sign of the transformed equations. As shown by G. A. Greenberg [4], this approach in the general case is not equivalent to that adopted in the eigenfunction method, but lead to satisfactory results. This is the main generalization in comparison with the eigenfunction method.

Integral transform methods are extensively used in the mechanics of deformable solids, and recently in structural mechanics.

The application of the integral transform method includes several main stages - the transition to mappings, finding the solution of the transformed equation and the transition to the original one, found at the second stage of the solution mapping. As a rule, at the first stage no difficulties arise. If the transformation is applied with respect to a part of the coordinates, then at the second stage it is necessary to solve differential equations, albeit of a lower dimension. But the main complications arise when finding the original solution. Such a problem does not exist in the finite integral transformation method, because here the finding of the original is reduced to the summation of series by the eigenfunctions of the operator of the boundary value problem, which is not a problem when applying numerical methods. True, a complication may arise here due to the slow convergence of the series, but in our time such complications are not significant. It should be noted that the procedure and properties of the finite integral transformation method are closely related to the procedure of the projection method. The essential difference is the fact that in the finite integral transformation method the eigenfunctions of the corresponding operator of the boundary value problem are used as basic functions, while in the projection method - the system of functions that is complete in the corresponding energy space. The eigenfunctions of the boundary value problem defined by the differential expression $\frac{\partial^2}{\partial\theta^2}$ and the boundary conditions of periodicity are a pair of functions sinθ, cosθ; sin2θ, cos2θ; sin3θ, cos3θ;...sin$n\theta$, cos$n\theta$, where each pair corresponds to one eigenvalue - n^2 [3]. These are functions of the Fourier series with respect to variable θ. This system of functions is orthogonal and must be normalized, namely considered in the form $\frac{1}{\sqrt{\pi}}\sin\theta$, $\frac{1}{\sqrt{\pi}}\cos\theta$; $\frac{1}{\sqrt{\pi}}\sin 2\theta$, $\frac{1}{\sqrt{\pi}}\cos 2\theta$; $\frac{1}{\sqrt{\pi}}\sin 3\theta$, $\frac{1}{\sqrt{\pi}}\cos 3\theta$,...

Two variants of finite-dimensional integral transforms are used - finite-dimensional integral transforms with basis functions

$$\frac{1}{\sqrt{\pi}}\sin\theta,\ \frac{1}{\sqrt{\pi}}\sin 2\theta,\ \frac{1}{\sqrt{\pi}}\sin 3\theta,\dots\ \frac{1}{\sqrt{\pi}}\sin m\theta,\dots\ \frac{1}{\sqrt{\pi}}\sin M\theta, \tag{30}$$

which is called sine transform; and the second finite integral transform with basic functions

$$\frac{1}{\sqrt{\pi}}\cos\theta,\ \frac{1}{\sqrt{\pi}}\cos 2\theta,\ \frac{1}{\sqrt{\pi}}\cos 3\theta,\dots\ \frac{1}{\sqrt{\pi}}\cos n\theta,\dots\ \frac{1}{\sqrt{\pi}}\cos N\theta, \tag{31}$$

which is called the cosine transform.

In specific calculations, the numbers M and N are selected so as to ensure the required accuracy of calculations.

Inverse transform (transition to original)

- for sine transform

$$\overset{M}{T}{}^{i}(\theta)=\sum_{m=1}^{M}\overset{m}{T}(\theta)\frac{1}{\sqrt{\pi}}\sin m\theta; \tag{32}$$

- for cosine transform

$$\overset{N}{T}{}^{i}(\theta)=\sum_{n=1}^{N}\overset{n}{T}(\theta)\frac{1}{\sqrt{\pi}}\cos n\theta. \tag{33}$$

Since the orthonormal basis of functions is used here, the reciprocal basis coincides with the basic basis and the difference between covariant and contravariant components disappears, so here the index m (or n) is located above the corresponding parts of an expression. The indices n and m vary from 1 to ∞, but in practice the finite number of members of the series are used, ie the index m varies from 1 to M, and the index n - from 1 to N to specify the calculation. To move to the mappings with respect to variable θ, the original equation (3) in the second-order derivatives is scalarly multiplied by $\frac{1}{\sqrt{\pi}}\sin m\theta$ - the transform kernel for finite integral sine transform and separately by $\frac{1}{\sqrt{\pi}}\cos n\theta$ - the transform kernel for the cosine transform. Since most of the components of equation (3) have no derivatives with respect to θ, their sine and cosine transforms are determined by scalar multiplication by $\frac{1}{\sqrt{\pi}}\sin n\theta$ and by $\frac{1}{\sqrt{\pi}}\cos m\theta$ correspondingly (without the operator $\frac{\partial^2}{\partial\theta^2}$). Resulting in:

$$\frac{\rho c}{\lambda_T}\cdot\frac{\partial \overset{m}{T}}{\partial t}=\frac{\partial^2 \overset{m}{T}}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \overset{m}{T}}{\partial r}+\frac{\partial^2 \overset{m}{T}}{\partial z^2}+\overset{m}{Q}_T, \tag{34}$$

$$\frac{\rho c}{\lambda_T}\cdot\frac{\partial \overset{n}{T}}{\partial t}=\frac{\partial^2 \overset{n}{T}}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \overset{n}{T}}{\partial r}+\frac{\partial^2 \overset{n}{T}}{\partial z^2}+\overset{n}{Q}_T. \tag{35}$$

It should be noted that these transforms primarily concern given influences, which include volumetric influences Q_T and influences on boundary surfaces marked with a hyphen, while the index n means sine transform of external influences, and m - cosine transformation.

The transformation of the differential operator $\frac{\partial^2}{\partial z^2}$ and the corresponding part of equation (3) is performed separately (including only that part of the equation that changes significantly in this case):

$$\left(\frac{\partial^2 T}{\partial \theta^2},\frac{1}{\sqrt{\pi}}\sin m\theta\right)=\int_0^{2\pi}\frac{1}{\sqrt{\pi}}\sin m\theta d\theta.$$

To transform the integral, the integration by parts is used twice (taking into account the second derivative):

$$\left(\frac{\partial}{\partial\theta}\left(\frac{\partial T}{\partial\theta}\frac{1}{\sqrt{\pi}}\sin m\theta d\theta\right)\right)=\left|\begin{array}{ll} dv=\frac{\partial}{\partial\theta}\left(\frac{\partial T}{\partial\theta}\right)d\theta;\ v=\frac{\partial T}{\partial\theta} & =u\cdot v\Big|_0^{2\pi}=0 \\ u=\frac{1}{\sqrt{\pi}}\sin m\theta;\ du=\frac{1}{\sqrt{\pi}}m\cdot\cos m\theta. & v\cdot du=\frac{\partial T}{\partial\theta}\cdot m\cos m\theta. \end{array}\right.$$

The component outside the sign of the integral is zero. The conditions of periodicity are fulfilled:

$$-\int_0^{2\pi}\frac{\partial T}{\partial\theta}m\frac{\cos m\theta}{\sqrt{\pi}}d\theta=\left|\begin{array}{l} dv=\frac{\partial T}{\partial\theta}d\theta,\ v=T, \\ u=\frac{m}{\sqrt{\pi}}\cos m\theta,\ du=\frac{m}{\sqrt{\pi}}\sin m\theta d\theta=0-\int_0^{2\pi}m^2\frac{\sin m}{\sqrt{\pi}d\theta}=-m^2\overset{m}{T}. \end{array}\right. \tag{36}$$

Similarly, n is obtained:

$$\left(\frac{\partial^2 T}{\partial\theta^2},\frac{1}{\sqrt{\pi}}\cos n\theta\right)=-n^2 T. \tag{37}$$

Taking into account the dimension reduction with respect to z, the reduced equations in the form of systems of equations of the second order in the spatial coordinate are obtained:

$$\frac{\rho c}{\lambda_T}\cdot\frac{\partial \overset{m}{T^\alpha}}{\partial t}=\frac{1}{R_0+x}\cdot\frac{\partial \overset{m}{T^\alpha}}{\partial x}+\frac{\partial^2 \overset{m}{T^\alpha}}{\partial x^2}-\frac{\overset{m}{T^\alpha}}{(R_0+x)^2}\cdot m^2-g^{\alpha i}\cdot b_{ji}\cdot g^{jk}\cdot b_{k\beta}\cdot \overset{m}{T^\beta}-\frac{g^{\alpha i}}{\lambda_T}[\overset{mN}{q_z}-\overset{m1}{q_z}]+\frac{\overset{m}{Q^\alpha}}{\lambda_T}, \tag{38}$$

$$\frac{\rho c}{\lambda_T}\cdot\frac{\partial \overset{n}{T^\alpha}}{\partial t}=\frac{1}{R_0+x}\cdot\frac{\partial \overset{n}{T^\alpha}}{\partial x}+\frac{\partial^2 \overset{n}{T^\alpha}}{\partial x^2}-\frac{n}{(R_0+x)^2}\cdot n^2-g^{\alpha i}\cdot b_{ji}\cdot g^{jk}\cdot b_{k\beta}\cdot \overset{n}{T^\beta}-\frac{g^{\alpha i}}{\lambda_T}[\overset{nN}{q_z}-\overset{n1}{q_z}]+\frac{\overset{n}{Q^\alpha}}{\lambda_T}. \tag{39}$$

Conclusions. The modified method of lines in combination with expansion on the circular coordinate into Fourier series allows the subsequent application of the method of discrete orthogonalization for the numerical solution of the given problems of thermoelasticity. The flexibility of the approach lies in the possibility of modeling any arbitrary boundary conditions. A convenient index form of writing reduced equations creates comfortable conditions for further programming in the FORTRAN and C++ algorithmic languages. At the stage of forming the reduced equations, it is possible to choose a coordinate of increased accuracy z or r, depending on the problem and the geometry of the studied objects.

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Сович Ю.В., Левківський Д.В., Янсонс М.О., Кошевий О.П., Пошивач Д.В.

ЧИСЕЛЬНО-АНАЛІТИЧНИЙ ПІДХІД ДО РОЗВ’ЯЗАННЯ ЗАДАЧ НЕСТАЦІОНАРНОЇ ТЕПЛОПРОВІДНОСТІ НЕТОНКОЇ КІЛЬЦЕВОЇ ПЛАСТИНИ

У даній роботі розглянуто перший етап розрахунку початково-граничної задачі нестаціонарної теплопровідності тіл циліндричної форми за допомогою модифікованого методу прямих, а саме зниження вимірності вихідних диференціальних рівнянь, початкових та граничних умов. Вихідні рівняння теплопровідності визначені в циліндричній системі координат в просторовій постановці. У якості об’єкта розглядається циліндричне тіло із співрозмірними габаритними розмірами. Даний напрям досліджень є актуальним, оскільки при розрахунку несучих елементів конструкцій на теплові впливи, першим етапом є визначення розподілу температурних полів. Граничні умови розглядаються як умови конвективного теплообміну, які за допомогою граничного переходу перетворюються на граничні умови першого та другого типів.

Зниження вимірності по просторовій координаті z виконується проекційним методом Бубнова-Гальоркіна-Петрова за допомогою локальних базисних функцій. Дані функції називають функції-«кришки», які пов’язані з прямими, що нанесені на область визначення поставленої задачі. Для зниження вимірності рівнянь по коловій координаті використовуються нормовані тригонометричні ряди. Всі перетворення виконуються в індексній формі. Крім диференціальних рівнянь, проекційним методом виконується зниженні вимірності початкових та граничних умов. У даній роботі визначено найбільш оптимальну форму запису редукованих рівнянь, що забезпечує легкість зниження вимірності вихідних диференціальних рівнянь. Початкові та граничні умови враховують вплив навколишнього середовища. Все це дає можливість поставити редуковану початково-граничну задачу для подальшого розрахунку чисельними скінченно-різницевими методами.

Ключові слова: теплопровідність, конвективний теплообмін, зниження вимірності, модифікований метод прямих, проекційний метод, редуковані рівняння, тригонометричні ряди, базисні функції.

Sovych Y.V., Levkivskyi D.V., Yansons M.O., Koshevyi O.P., Poshyvach D.V.

NUMERICAL-ANALYTICAL APPROACH TO SOLVING PROBLEMS OF NON-STATIONARY THERMAL CONDUCTIVITY OF A NON-THIN ANNULAR PLATE

This paper considers the first stage of calculating the initial boundary value problem of non-stationary thermal conductivity of cylindrical bodies using a modified method of lines, namely dimension reduction of the original differential equations, initial and boundary conditions. The original equations of thermal conductivity are defined in a cylindrical coordinate system in a spatial setting. An object is a cylindrical body with commensurate dimensions. This area of research is relevant, because when calculating the load bearing elements of structures to thermal effects, the first step is to determine the distribution of temperature fields. Boundary conditions are considered as conditions of convective heat transfer, which by means of boundary transition are transformed into boundary conditions of the first and second types.

Dimension reduction with respect to spatial coordinate z is performed by the Bubnov-Galorkin-Petrov projection method using local basis functions. These functions are called "cover" functions, which are related to the lines drawn on the domain of the task. Normalized trigonometric series are used to reduce the dimensionality of the equations with respect to circular coordinate. All transformations are performed in index form. In addition to differential equations, the projection method reduces the dimensionality of the initial and boundary conditions. In this paper, the most optimal form of writing reduced equations is determined, which provides the ease of reducing the dimensionality of the original differential equations. Initial and boundary conditions take into account the impact of the environment. All this makes it possible to set a reduced initial boundary value problem for further calculation by numerical finite difference methods.

Keywords: thermal conductivity, convective heat transfer, dimension reduction, modified method of lines, projection method, reduced equations, trigonometric series, basic functions.

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Сович Ю.В, Левківський Д.В., Янсонс М.О., Кошевий О.П., Пошивач Д.В. **Чисельно-аналітичний підхід до розв'язання задач нестаціонарної теплопровідності нетонкої кільцевої пластини** // Опір матеріалів і теорія споруд: наук.-тех. збірн. – К.: КНУБА, 2024. – Вип. 112. – С. 185-194.
У роботі розглянуто застосування модифікованого методу прямих для зниження вимірності диференціальних рівнянь нестаціонарної теплопровідності у циліндричній системі координат. Зниження вимірності вихідних рівнянь виконується за допомогою проекційного методу по координаті z. Для цього використовуються локальні базисні функції. Редуковані рівняння доповнюються редукованими початковими та граничними умовами. По коловій координаті зниження вимірності виконується за допомогою нормованих тригонометричних рядів. У результаті отримано редуковані диференціальні рівняння, початкові та граничні умови, що залежать від радіальної та часової координати. Дані рівняння підготовлені для подальшого розрахунку скінченно-різницевими чисельними методами.розв'язків редуковані рівняння доповнються початковими та граничними умовами.
Іл. 2. Бібліог. 10 назв.

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Sovych Y.V., Levkivskyi D.V., Yansons M.O., Koshevyi O.P., Poshyvach D.V. **Numerical-analytical approach to solving problems of non-stationary thermal conductivity of a non-thin annular plate** // Strength of Materials and Theory of Structures: Scientific-and-technical collected articles – Kyiv: KNUBA, 2024. – Issue 112. – P. 185-194.
In this paper, the application of a modified method of lines for dimension reduction of differential equations of nonstationary thermal conductivity in a cylindrical coordinate system is considered. Dimension reduction of the original equations is implemented using the projection method with respect to variable z. Local basic functions are used for this purpose. Reduced equations are supplemented by reduced initial and boundary conditions. Dimension reduction with respect to circular coordinate is carried out by means of normalized trigonometric series. As a result, reduced differential equations, initial and boundary conditions, which depend on the radial and temporal coordinates, are obtained. These equations are prepared for further calculation by finite difference numerical methods.
Figs. 2. Refs. 10.

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