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INVESTIGATION OF THE SECOND MAIN PROBLEM OF ELASTICITY FOR A LAYER WITH *N* CYLINDRICAL INCLUSIONS

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The analysis of the stress-strain state for a layer with N longitudinal cylindrical inclusions, when the specified displacements are given, has been carried out. The solution of the spatial problem has been obtained by the generalized Fourier method with respect to the system of Lame's equation in cylindrical coordinates associated with inclusions and Cartesian coordinates associated with layer boundaries. Infinite systems of linear algebraic equations obtained by satisfying the boundary conditions and conjugation conditions of a layer with inclusions have been solved by the reduction method. As a result, stresses have been obtained at various points of the elastic body. A numerical study compares the options of the layer stress-strain state with one and three cylindrical elastic inclusions.

Keywords: layer with cylindrical inclusions, Lame's equation, generalized Fourier method, infinite systems of linear algebraic equations.

Introduction

Determination of the stress-strain state of a body is the main task in the design of various kinds of structures, buildings, and underground communications. When designing the listed objects, computational schemes, where a layer densely reinforced with longitudinal bars becomes the main design element, are often encountered. At present, the mutual influence of these bars has been little studied, and there is no technique for high-precision solution of such computational schemes at all.

However, there are works close to the task at hand in the scientific literature. Thus, in [1 – 4], problems are solved for a layer with cavities perpendicular to its boundaries.

Stationary problems of diffraction of elastic waves with cylindrical cavities parallel to the layer boundaries are considered in [5 – 7], where the generalized Fourier method is used in combination with the image method.

For elastic bodies with several limiting surfaces, the generalized Fourier method [8], which is based on the addition theorems for the basic solutions of Lame's equation, is used.

Using this method, the following problems have been solved: for a layer with a spherical cavity [9], for a half-space with longitudinal cylindrical cavities [10 – 11], for a cylinder with cylindrical cavities and inclusions [12 –

16]. Problems for a layer with one or several cylindrical cavities are considered in [17 – 20], for a layer with an elastic inclusion—in [21, 22].

For a layer with several cylindrical inclusions and displacements specified on limiting surfaces, there are no ready-made algorithms in the spatial version, so the problem of calculating such problems is urgent.

The purpose of this work is:

– development of an analytical-numerical method for calculating the second main problem of elasticity (displacements are specified on all limiting surfaces) for a layer with N cylindrical inclusions that are parallel to the surfaces of the layer and to each other;

– to analyze the influence of inclusions on the stress state of the layer, as well as on each other.

Problem statement

There are N circular cylindrical inclusions with radius R_p in an elastic homogeneous layer (Fig. 1).

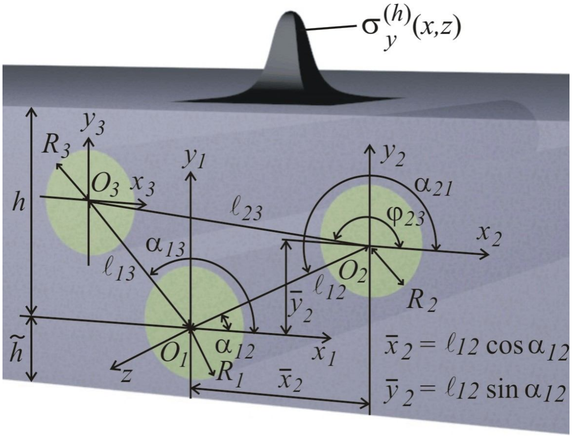


Fig. 1. Layer with cylindrical inclusions

The inclusion will be considered in local cylindrical coordinate systems (ρ_p, φ_p, z), the layer will be considered in the Cartesian coordinate system (x, y, z) connected to the inclusion coordinate system with the number p = 1. The layer boundaries are located at the distance $y = h$ and $y = -\tilde{h}$.

It is necessary to find a solution of Lame's equation $\Delta\vec{U}+(1-2\sigma)^{-1}\nabla div\vec{U}=0$ provided that displacements are specified at the boundaries of the layer:

$$\vec{U}_0(x,z)_{|y=h}=\vec{U}_h^0(x,z)\,,\ \vec{U}_0(x,z)_{|y=-\tilde{h}}=\vec{U}_{\tilde{h}}^0(x,z)\,,$$

where $\vec{U}_0$ —displacements in the layer;

$$\vec{U}_h^0(x,z)=U_x^{(h)}\vec{e}_x+U_y^{(h)}\vec{e}_y+U_z^{(h)}\vec{e}_z \\ \vec{U}_{\tilde{h}}^0(x,z)=U_x^{(\tilde{h})}\vec{e}_x+U_y^{(\tilde{h})}\vec{e}_y+U_z^{(\tilde{h})}\vec{e}_z, \qquad (1)$$

known functions, which will be considered to be as rapidly decreasing to zero at long distances from the origin along the x and z coordinates.

At the boundaries of the contact between the layer and the inclusions, the specified matching conditions

$$\vec{U}_0(\varphi_p,z)\big|_{\rho_p=R_p}=\vec{U}_p(\varphi_p,z)\big|_{\rho_p=R_p}, \qquad (2)$$

$$F\vec{U}_0(\varphi_p,z)\big|_{\rho_p=R_p}=F\vec{U}_p(\varphi_p,z)\big|_{\rho_p=R_p}, \qquad (3)$$

where $\vec{U}_p$ —displacements in cylindrical inclusion with number p; $F\vec{U}=2G[\frac{\sigma}{1-2\sigma}\vec{n}\,\mathrm{div}U+\frac{\partial}{\partial n}\vec{U}+\frac{1}{2}(\vec{n}\times rot\vec{U})]$; $G=\frac{E}{2(1+\sigma)}$; σ, E - Poison's ratio and elasticity modulus.

Problem solving

Let us choose the basic solutions of Lame's equation for the indicated coordinate systems in the form [8]:

$$\vec{u}_k^{\pm}(x,y,z;\lambda,\mu)=N_k^{(d)}e^{i(\lambda z+\mu x)\pm\gamma y}; \\ \vec{R}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\varphi)}; \qquad (4) \\ \vec{S}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}\left[(\mathrm{sign}\,\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\varphi)}\right], k=1,2,3;$$

$$N_1^{(d)}=\frac{1}{\lambda}\nabla;\; N_2^{(d)}=\frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)}+\frac{1}{\lambda}\nabla(y\cdot);\; N_3^{(d)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(1)}\cdot\right);\; N_1^{(p)}=\frac{1}{\lambda}\nabla;$$

$$N_2^{(p)}=\frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right)+4(\sigma-1)\left(\nabla-\vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];\; N_3^{(p)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(2)}\cdot\right),$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$, $-\infty<\lambda,\mu<\infty$; $I_m(x)$, $K_m(x)$ —modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, k=1, 2, 3—the internal and external solutions of Lame's equation for the cylinder respectively; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ —solutions of Lame's equation for the layer.

We represent the solution to the problem in the form:

$$\vec{U}_0=\sum_{p=1}^{N}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}^{(p)}(\lambda)\cdot\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)d\lambda+ \\ +\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)+\tilde{H}_k(\lambda,\mu)\cdot\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)\right)d\mu d\lambda, \qquad (5)$$

$$\vec{U}_p=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{k,m}^{(p)}(\lambda)\cdot\vec{R}_{k,m}(\rho_p,\varphi_p,z;\lambda)d\lambda, \qquad (6)$$

where $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$, $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$ and $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ — basic solutions, which are given by formulas (4), and unknown functions $H_k(\lambda,\mu)$, $\tilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(p)}(\lambda)$, and $A_{k,m}^{(p)}(\lambda)$ should be found based on boundary conditions and matching conditions.

In order to pass the basic solutions between coordinate systems, we use the formulas [20]:

- to go from solutions $\vec{S}_{k,m}$ of a cylindrical coordinate system to solutions of the layer $\vec{u}_k^{(-)}$ (when $y>0$) and $\vec{u}_k^{(+)}$ (when $y<0$)

$$\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)=\frac{(-i)^m}{2}\int\limits_{-\infty}^{\infty}\omega_{\mp}^m\cdot e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\vec{u}_k^{(\mp)}\cdot\frac{d\mu}{\gamma},\ k=1,3,$$

$$\vec{S}_{2,m}(\rho_p,\varphi_p,z;\lambda)=\frac{(-i)^m}{2}\int\limits_{-\infty}^{\infty}\omega_{\mp}^m\cdot\left((\pm m\cdot\mu-\frac{\lambda^2}{\gamma}\pm\lambda^2\bar{y}_p)\vec{u}_1^{(\mp)}\mp\lambda^2\vec{u}_2^{(\mp)}\pm\right. \tag{7}$$

$$\left.\pm4\mu(1-\sigma)\vec{u}_3^{(\mp)}\right)\cdot\frac{e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}d\mu}{\gamma^2},$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$, $\omega_{\mp}(\lambda,\mu)=\frac{\mu\mp\gamma}{\lambda}$, $m=0,\pm1,\pm2,\ldots$;

- for the transition from solutions $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ of the layer to solutions $\vec{R}_{k,m}$ of the cylindrical coordinate system:

$$\vec{u}_k^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}\left(i\cdot\omega_{\mp}\right)^m\vec{R}_{k,m},\ k=1,3,$$

$$\vec{u}_2^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}\left[\left(i\cdot\omega_{\mp}\right)^m\cdot\lambda^{-2}\left((m\cdot\mu+\bar{y}_p\cdot\lambda^2)\cdot\vec{R}_{1,m}\pm\gamma\cdot\vec{R}_{2,m}+4\mu(1-\sigma)\vec{R}_{3,m}\right)\right], \tag{8}$$

where $\vec{R}_{k,m}=\vec{\tilde{b}}_{k,m}(\rho_p,\lambda)\cdot e^{i(m\varphi_p+\lambda z)}$; $\vec{\tilde{b}}_{1,n}(\rho,\lambda)=\vec{e}_\rho\cdot I_n'(\lambda\rho)+i\cdot I_n(\lambda\rho)\cdot\left(\vec{e}_\varphi\frac{n}{\lambda\rho}+\vec{e}_z\right)$;

$\vec{\tilde{b}}_{2,n}(\rho,\lambda)=\vec{e}_\rho\cdot\left[(4\sigma-3)\cdot I_n'(\lambda\rho)+\lambda\rho I_n''(\lambda\rho)\right]+\vec{e}_\varphi i\cdot m\left(I_n'(\lambda\rho)+\frac{4(\sigma-1)}{\lambda\rho}I_n(\lambda\rho)\right)+\vec{e}_z i\lambda\rho I_n'(\lambda\rho)$;

$\vec{\tilde{b}}_{3,n}(\rho,\lambda)=-\left[\vec{e}_\rho\cdot I_n(\lambda\rho)\frac{n}{\lambda\rho}+\vec{e}_\varphi\cdot i\cdot I_n'(\lambda\rho)\right]$; $\vec{e}_\rho$, $\vec{e}_\varphi$, $\vec{e}_z$ - unitary vectors of a cylindrical coordinate system;

- for passing from solutions of the cylinder with number p to solutions of the cylinder with number q:

$$\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)=\sum_{n=-\infty}^{\infty}\vec{b}_{k,pq}^{mn}(\rho_q)\cdot e^{i(n\varphi_q+\lambda z)},\ k=1,2,3\ , \tag{9}$$

$$\vec{b}_{1,pq}^{mn}(\rho_q)=(-1)^n \tilde{K}_{m-n}(\lambda \ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot \vec{\tilde{b}}_{1,n}(\rho_q,\lambda),$$

$$\vec{b}_{3,pq}^{mn}(\rho_q)=(-1)^n \tilde{K}_{m-n}(\lambda \ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot \vec{\tilde{b}}_{3,n}(\rho_q,\lambda),$$

$$\vec{b}_{2,pq}^{mn}(\rho_q)=(-1)^n \Big\{\tilde{K}_{m-n}(\lambda \ell_{pq})\cdot \vec{\tilde{b}}_{2,n}(\rho_q,\lambda)-\frac{\lambda}{2}\ell_{pq}\cdot$$

$$\cdot\Big[\tilde{K}_{m-n+1}(\lambda \ell_{pq})+\tilde{K}_{m-n-1}(\lambda \ell_{pq})\Big]\cdot \vec{\tilde{b}}_{1,n}(\rho_q,\lambda)\Big\}\cdot e^{i(m-n)\alpha_{pq}},$$

where α_{pq} - the angle between the x_p axis and the segment ℓ_{qp}; $\tilde{K}_m(x)=(sign(x))^m\cdot K_m(|x|)$.

The distance and angle between the parallel displaced cavities shall be calculated by the formulas:

$$L_{pq}=\left|\begin{array}{l}\sqrt{L_{1p}^2+L_{1q}^2-2\cdot L_{1p}\cdot L_{1q}\cdot\cos(\alpha_{1q}-\alpha_{1p})},\ at\ \alpha_{1q}\ge\alpha_{1p}\\ \sqrt{L_{1p}^2+L_{1q}^2-2\cdot L_{1p}\cdot L_{1q}\cdot\cos(\alpha_{1p}-\alpha_{1q})},\ at\ \alpha_{1q}<\alpha_{1p}\end{array}\right.,$$

$$\alpha_{pq}=\left|\begin{array}{l}\alpha_{1p}-\arcsin\big(L_{1q}\cdot\sin(\alpha_{1q}-\alpha_{1p})/L_{pq}\big)+\pi,\ at\ \alpha_{1q}\ge\alpha_{1p}\\ \alpha_{1p}-\arcsin\big(L_{1q}\cdot\sin(\alpha_{1q}-\alpha_{1p})/L_{pq}\big)-\pi,\ at\ \alpha_{1q}<\alpha_{1p}\end{array}\right..$$

In order to fulfill the boundary conditions at the layer boundaries, we are rewritten the vectors $\vec{S}_{k,m}$ in (5), using the transition formulas (7), in the Cartesian coordinate system through the basic solutions $\vec{u}_k^{(-)}$ for $y=h$ and for $y=-\tilde{h}$. We equate the obtained vectors with the given $\vec{U}_h^0(x,z)$ when $y=h$ and $\vec{U}_{\tilde{h}}^0(x,z)$ when $y=-\tilde{h}$ (1), represented through double Fourier integral.

From the obtained equations, we find the functions $H_k(\lambda,\mu)$ and $\tilde{H}_k(\lambda,\mu)$, using $B_{k,m}^{(p)}(\lambda)$.

In order to satisfy the matching conditions of the layer and the inclusion p in the displacements (2), we rewrite them in solutions $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ in $\vec{U}_0(\varphi_p,z)\big|_{\rho_p=R_p}$ in terms of basic solutions $\vec{R}_{k,m}(\rho_p,\varphi_p,z;\lambda)$ (8), and also for each inclusion $\neq p$, we rewrite the solutions $\vec{S}_{k,m}(\rho_q,\varphi_q,z;\lambda)$, using the solutions $\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)$, according to formulas (9). So we get three equations for each inclusion in the displacements. Applying the stress operator to each obtained expression, we can write three more equations for each inclusion in the form of stresses (2). Thus, we obtain $N\cdot 6$ infinite equations with $(N+1)\cdot 6$ unknowns $H_k(\lambda,\mu)$, $\tilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(p)}(\lambda)$. If we now exclude from these equations $H_k(\lambda,\mu)$, $\tilde{H}_k(\lambda,\mu)$ found earlier, using

$B_{k,m}^{(p)}(\lambda)$, get rid of the series in m and integrals in λ, then we get $N \cdot 6$ infinite systems of linear algebraic equations of the second kind for determining the unknowns $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(p)}(\lambda)$.

The determinant of this system of equations coincides with [22].

These systems can be solved by the reduction method, and there is a convergence of approximate solutions to the exact one. As a result, we find the unknown $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(p)}(\lambda)$. Now we substitute $B_{k,m}^{(p)}(\lambda)$ in the expressions for $H_k(\lambda,\mu)$ and $\tilde{H}_k(\lambda,\mu)$, this will define all the unknown problems.

Numerical studies of the stress state

We have a homogeneous isotropic layer (ABS plastic) σ_0 = 0.38, E_0 = 1,700 N/mm^2, which has three longitudinal cylindrical inclusions (steel) $\sigma_1 = \sigma_2 = \sigma_3$ = 0.21, $E_1 = E_2 = E_3$ =200,000 N/mm^2. Geometric characteristics of the section: $R_1 = R_2 = R_3$ = 10 mm, h = 20mm, $\tilde{h}$ = 40 mm, $L_{12} = L_{13}$ = 30 mm, $\alpha_{12} = 0$, $\alpha_{13} = \pi$.

Displacements are set on the upper boundary $U_y^{(h)}(x,z) = -10^8 \cdot (z^2+10^2)^{-2} \cdot (x^2+10^2)^{-2}$, $U_x^{(h)} = U_z^{(h)} = 0$, there are no displacements on the lower boundary of the layer $U_x^{(\tilde{h})} = U_y^{(\tilde{h})} = U_z^{(\tilde{h})} = 0$.

The infinite system of equations was reduced to a finite one—m=6. The integrals are calculated using the Philo quadrature formulas (for oscillating functions) and Simpson (for functions without oscillations). The accuracy of the boundary conditions at the specified values of geometric parameters is 10^{-2}.

In comparison, the problem for a layer with one cylindrical inclusion has been calculated [22].

Figure 2 shows the stress state on the cylindrical mating surface of the layer and the first inclusion (in the body of the layer), at z = 0, MPa.

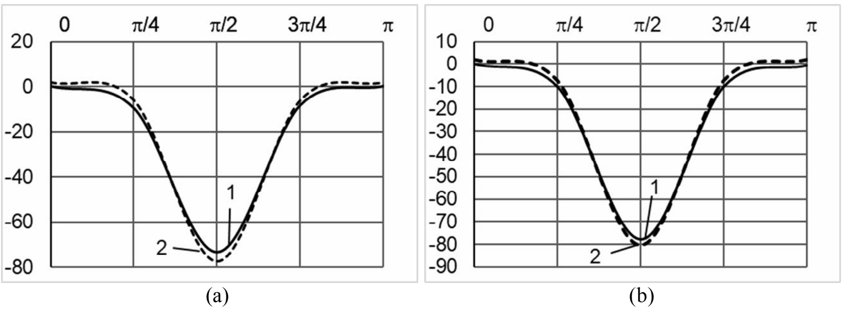


Fig. 2. Stress state on the mating surface (in the body of the layer):
(a) – σ_φ; (b) – σ_z; 1 – three inclusions; 2 – one inclusion

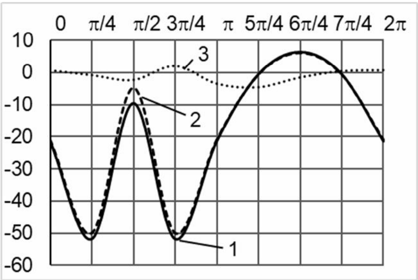
With an increase in the number of longitudinal bars, the maximum stresses in the layer body decrease slightly (Fig. 2).

The maximum stresses occur in the upper part of the layer. In the lower part of the layer, the stresses are relatively small; therefore, they are not shown in Figure 2.

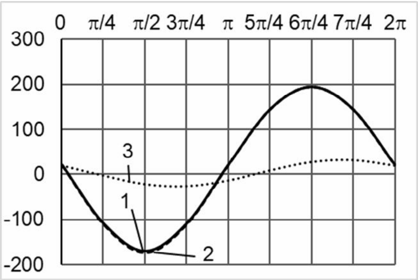
The stresses in the body of inclusions are shown in Figure 3.

The maximum negative stresses σ_φ (Fig. 3 (a)) arise at $\varphi = \pi/4$ and $\varphi = 3\pi/4$ in the upper part of the inclusion.

The largest stresses σ_z are positive, the maximum values of which arise in the lower part of the inclusion at $\varphi = 6\pi/4$ (Fig. 3 (b)). Negative stress values σ_z appear in the upper part of the inclusion at $\varphi = \pi/2$.



(a)



(b)

Fig. 3. Stress state on the mating surface (in the body of the layer):
(a) – σ_φ; (b) – σ_z; 1 – in the body of the first inclusion; 2 – option with one inclusion; 3 – in the body of the second and the third inclusions

The greatest tangent stresses $\tau_{\rho\varphi}$ on the mating surface arise in its upper part at $\varphi = \pi/4$ and $\varphi = 3\pi/4$.

Conclusions

Based on the generalized Fourier method, an analytical-numerical algorithm for calculating the second main spatial problem of elasticity for the layer with N cylindrical inclusions that are parallel to the layer surfaces has been developed. The problem is reduced to a set of infinite systems of linear algebraic equations.

On the strength of numerical studies of the algebraic system, it can be argued that the solution of this system can be found with any degree of accuracy by the reduction method. This is confirmed by the high accuracy of fulfilling the boundary conditions.

A comparative analysis of the layer stress state with one and three inclusions has been carried out. When the inclusions are located close, an increase in stresses σ_φ in their bodies is revealed.

Further studies in this direction are relevant for a layer lying on an elastic foundation (conjugate with a half-space) and having several longitudinal cylindrical inclusions.

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ДОСЛІДЖЕННЯ ДРУГОЇ ОСНОВНОЇ ЗАДАЧІ ТЕОРІЇ ПРУЖНОСТІ ДЛЯ ШАРУ З *N* ЦИЛІНДРИЧНИМИ ВКЛЮЧЕННЯМИ

При проектуванні конструкцій у вигляді армованого шару доводиться стикатися з ситуацією, коли стержні армування розташовані близько один до одного. У цьому випадку зростає вплив їх один на одного. Для отримання напружено-деформованого стану в зоні контакту шару і включення необхідно мати метод, який би дозволяв отримати результат з високою точністю.

У цій роботі запропоновано аналітико-чисельний підхід до вирішення просторової задачі теорії пружності для шару з заданою кількістю поздовжніх циліндричних включень і заданих на межах шару переміщеннях.

Розв’язок задачі отримано на основі узагальненого методу Фур'є щодо системи рівнянь Ламе в локальних циліндричних координатах, пов'язаних з включеннями і декартових координатах, пов'язаних з межами шару. Нескінченні системи лінійних алгебраїчних рівнянь, які отримані в результаті задоволення граничних умов і умов сполучення шару з включеннями, розв’язано методом редукції. В результаті отримані переміщення і напруження в різних точках розглянутого середовища. При порядку системи рівнянь 6, точність виконання граничних умов склала 10^{-2} для значень від 0 до 1.

Чисельні дослідження алгебраїчної системи рівнянь дають підстави стверджувати, що її рішення може бути з будь-яким ступенем точності знайдено методом редукції, що підтверджується високою точністю виконання граничних умов.

У чисельному аналізі порівнювалися варіанти шару з одним і з трьома включеннями. Результат показав, що близьке розташування стержнів армування збільшує напруження на поверхні цих включень. Також були отримані значення напружень на поверхнях контактів шару з включеннями.

Запропонований алгоритм розв’язання можна використовувати при проектуванні конструкцій, розрахунковою схемою яких є шар з поздовжніми циліндричними включеннями і заданими на межах шару переміщеннях.

Ключові слова: шар з циліндричними включеннями, рівняння Ламе, узагальнений метод Фур'є, нескінченні системи лінійних алгебраїчних рівнянь.

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INVESTIGATION OF THE SECOND MAIN PROBLEM OF ELASTICITY FOR A LAYER WITH N CYLINDRICAL INCLUSIONS

When designing structures in the form of a reinforced layer, one has to deal with the situation when the reinforcement bars are located close to each other. In this case, their influence on each other increases. In order to obtain the stress-strain state in the contact zone of the layer and the inclusion, it is necessary to have a method that would allow obtaining a result with high accuracy.

In this work, an analytical-numerical approach to solving the spatial problem of the theory of elasticity for a layer with a given number of longitudinal cylindrical inclusions and displacements given at the boundaries of the layer has been proposed.

The solution of the problem has been obtained by the generalized Fourier method with respect to the system of Lame's equation in local cylindrical coordinates associated with inclusions and Cartesian coordinates associated with layer boundaries. Infinite systems of linear algebraic equations obtained by satisfying the boundary conditions and conjugation conditions of a layer with inclusions have been solved by the reduction method. As a result, displacements and stresses have been obtained at different points of the considered medium. When the order of the system of equations is 6, the accuracy of fulfilling the boundary conditions was 10^{-2} for values from 0 to 1.

Numerical studies of the algebraic system of equations give grounds to assert that its solution can be found with any degree of accuracy by the reduction method, which is confirmed by the high accuracy of fulfilling the boundary conditions.

In the numerical analysis, variants of the layer with 1 and 3 inclusions have been compared. The result has shown that close placement of reinforcement bars increases stresses σ_{φ} on the surface of these inclusions. The values of stresses on the layer contact surfaces with inclusions have also been obtained.

The proposed solution algorithm can be used in the design of structures, the computational scheme of which is the layer with longitudinal cylindrical inclusions and displacements specified at the layer boundaries.

Keywords: layer with cylindrical inclusions, Lame's equations, generalized Fourier method, infinite systems of linear algebraic equations.

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Мірошніков В.Ю., Денисова Т.В. **Дослідження другої основної задачі теорії пружності для шару з *N* циліндричними включеннями** // Опір матеріалів і теорія споруд: наук.-техн. збірн. – К.: КНУБА, 2021. – Вип. 106. – С. 156-166.

Досліджено напружено – деформований стан шару з декількома циліндричними включеннями, коли на межах шару задані переміщення.

Табл. 0. Іл. 3. Бібліогр. 22 назв.

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Miroshnikov V.Yu., Denisova T.V. **Investigation of the second main problem of elasticity for a layer with *n* cylindrical inclusions** // Strength of Materials and Theory of Structures: Scientific and technical collected articles. - Kyiv: KNUBA, 2021. - Issue 106. - P. 156-166.

The stress - strain state of a layer with several cylindrical inclusions when displacements are set at the layer boundaries is investigated.

Tabl. 0. Fig. 3. Ref. 22.

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Мирошников В.Ю., Денисова Т.В. **Исследование второй основной задачи теории упругости для слоя с *N* цилиндрическими включениями** // Сопротивление материалов и теория сооружений: науч.-техн. сборн. - К.: КНУСА, 2021. - Вып. 106. - С. 156-166.

Исследовано напряженно - деформированное состояние слоя с несколькими цилиндрическими включениями, когда на границах слоя заданы перемещения.

Табл. 0. Ил. 5. Библиогр. 18 назв.

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