The influence of functional heterogeneity of the material on mechanical oscillations of piezoelements at non-stationary electrical loads is investigated. Within the assumption of the corresponding to body physical properties functional distribution of material characteristics by thickness of piezoelectric element, a unified system of solving equations was obtained to describe the thickness fluctuations of piezoelectric plates, thick-walled cylinders and balls. For controlling of accuracy, the calculation is performed with explicit and implicit difference scheme. It was established that influence of functional heterogeneity within one material is reduced to decrease of wave propagation velocity and the amplitude of oscillations up to 2-3%. We do a conclusion that heterogeneity effect on the oscillations of the piezoelement is low, so the material characteristics can be averaged by thickness.

**Key words:** piezoceramics, electroelasticity, functional-gradient material, non-stationary oscillations.

**Introduction.** At the moment, the calculation of the dynamic behavior of piezoelectric bodies faces the problem of matching theoretical results with practical ones. During the operation of piezoelement, possible deviations of the characteristic of the operating mode from the predicted ones can be up to 20% [12]. Such deviations are possible as a result of differences in the actual material characteristics from those declared by the manufacturer, non-homogeneity of the material, inconsistency of the operating conditions with the predicted ones (the influence of the environment is not taken into account, the fastening and loading conditions are not correctly implemented or programmed, the influence of the body and protective shells is not taken into account) etc.

Currently, the oscillations of piezoelectric bodies in linear assumption are widely presented in the literature. In particular, modern methods and approaches to calculating of dynamic behavior of electroelastic bodies, interaction with acoustic and thermal fields, working standards for determining material characteristics are given in [6]. The main types, geometry, operating modes and features of operation of the most common piezoelements are described in [1].

During calculations there are usually accepted determined experimentally averaged values of material characteristics. At the same time, at analyzing the
manufacturing and polarization process of piezoelectric bodies \[1\], in particular thick-walled ones, the heterogeneity of material characteristics by thickness is obvious.

In the latest publications it is noted about the wide possibilities and prospects of practical application of constructive elements, in particular sensors, actuators, energy harvesters, made of so-called functional-gradient piezoceramics \[2\]. Such elements, thanks to the composition of piezoactive and piezopassive materials, allow to strengthen the advantages of piezoceramic elements. This is due to the increasing interest and relevance of dynamic behavior theoretical studies of functional-gradient piezoelectrics under various variants of their layout and electro-mechanical load. In \[10\], the latest technology for the manufacturing of a functional gradient piezoelectric film based on a polymer and barium titanate is described. In \[2, 7, 9\] the separating variables method is used for free vibrations and active control of rectangular plates \[7, 9\] and for the functional-gradient piezoceramics spherical rotating shell \[2\]. In \[3\], a cantilever nanobeam made of flexoelectric material is considered. The gradient dependence is given in the form of an exponent. There is also introduced the flexoelectric coefficient \(\mu\), which allows to take into account large deflections. In \[5\] there is investigated a rectangular three-layer plate with outer piezoelectric layers and an inner passive functional-gradient layer.

This work presents a solution of the problem of the effect of material functional heterogeneity on non-stationary thickness fluctuations of thick-walled piezoelectric elements. At this it is considered that the specified heterogeneity corresponds to the geometric features and the procedure of manufacturing the bodies considered in the work. The problem is solved using the universal approach for studying the vibrations of flat layers, cylinders and spheres, proposed in publications \[4, 8\]. The calculations are carried out using explicit and implicit difference schemes for controlling the results accuracy.

**Formulation of the problem.** Thickness fluctuations of flat bodies \((N=0)\), cylinders \((N=1)\), and balls \((N=2)\) are described by the equation of motion \[4, 8\]:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{N}{r} (\sigma_r - \sigma_\theta) = \rho \frac{\partial^2 u_z}{\partial t^2},
\]

and Maxwell’s quasi-static equation for electrical variables

\[
\frac{\partial D_r}{\partial r} + \frac{N}{r} D_r = 0,
\]

where \(r\) is thickness coordinate.

Material relations with thickness polarization

\[
\sigma_r = e_{33}^E \frac{\partial u}{\partial r} + N \frac{c_{13}^E}{r} u + e_{33} \frac{\partial \varphi}{\partial r},
\]

\[
\sigma_\theta = c_{11}^E \frac{\partial u}{\partial r} + \frac{N}{r} \left( c_{11}^E - \frac{1}{2} (N-1) (c_{11}^E - c_{13}^E) \right) u + e_{13} \frac{\partial \varphi}{\partial r},
\]

\[
D_r = e_{33} \frac{\partial u}{\partial r} + \frac{N e_{31}}{r} u - e_{33}^S \frac{\partial \varphi}{\partial r}.
\]
Further in the work, we will assume that all material characteristics are known functions of the thickness coordinate:
\[ c^E_\ast = c^E_\ast (r), \quad e_\ast = e_\ast (r), \quad \varepsilon^S_\ast = \varepsilon^S_\ast (r). \] (4)

We introduce dimensionless in such a way that the original equations do not change their form:
\[
\bar{r} = \frac{r}{h}, \quad \bar{t} = \frac{t}{t_h}, \quad \bar{u} = \frac{u}{h}, \quad \bar{\sigma}_\ast = \frac{\sigma_\ast}{c_0^0}, \quad \bar{\varphi} = \frac{\varphi}{\sqrt{\varepsilon^S_33 c_0^0}}, \quad \bar{D}_r = \frac{D_r}{\sqrt{c_0^0 \varepsilon^S_33}},
\]
\[
\bar{c}_\ast = \frac{c^E_\ast}{c_0^0}, \quad \bar{e}_\ast = \frac{e_\ast}{\sqrt{c_0^0 \varepsilon^S_33}}, \quad \bar{\varepsilon}_{33} = 1,
\]
where \( h \) is a layer thickness, \( t_h = \frac{D}{c_0} \), \( c_0^0 = c^E_33 + \frac{e^2_33}{\varepsilon^S_33} \). In the future, we omit the dimensionless sign (macro) and only dimensionless variables are used.

We substitute (3) into (1), (2) taking into account the functional dependence of material constants from the coordinate \( r \)
\[
\rho \frac{\partial^2 u}{\partial t^2} = c_{33} \frac{\partial^2 u}{\partial r^2} + \left( c'_{33} + N \frac{c_{33}}{r} \right) \frac{\partial u}{\partial r} + \left( N \frac{c_{13}'}{r} - N \frac{c_{13}}{r^2} + N^2 \frac{c_{13} - a}{r^2} \right) u + \frac{e_{33}}{\varepsilon^S_33} \frac{\partial^2 \varphi}{\partial r^2} + \left( e'_{33} + N \frac{e_{33} - e_{31}}{r} \right) \frac{\partial \varphi}{\partial r},
\]
\[
e_{33} \frac{\partial^2 u}{\partial r^2} + \left( e'_{33} + N \frac{e_{31} + e_{33}}{r} \right) \frac{\partial u}{\partial r} + \left( N \frac{e_{31}'}{r} + N (N - 1) \frac{e_{31}}{r^2} \right) u - \frac{e_{33}}{\varepsilon^S_33} \frac{\partial^2 \varphi}{\partial r^2} = 0,
\] (5)
\[
e_{33} \frac{\partial^2 u}{\partial r^2} + \left( e'_{33} + N \frac{e_{33}}{r} \right) \frac{\partial u}{\partial r} + \left( N \frac{e_{33}}{r} + N (N - 1) \frac{e_{31}}{r^2} \right) u - \frac{e_{33}}{\varepsilon^S_33} \frac{\partial^2 \varphi}{\partial r^2} = 0,
\] (6)
where \( a = c_{11} - (N - 1) (c_{11} - c_{23}) / 2 \). The dash denotes the function derivative along the thickness coordinate.

Equations (5)(6) are supplemented with mechanical boundary conditions
\[ u(R_0, t) = 0, \quad \sigma_{rr}(R_1, t) = P_1(t) \] (7)
and electrical boundary conditions
\[ \varphi(R_0) = 0, \quad \varphi(R_1) = V(t), \] (8)
where \( r_1 = R_0 \) is a coordinate of bottom surface of a layer \( r_m = R_1 \) is a coordinate of top surface. Relations (7) (9) model the rigid fixation of the lower boundary surface of the piezoelement and zero electric potential on it.

We write down the initial conditions as follows
\[ u|_{t=0} = U_0(r), \quad \frac{du}{dt}|_{t=0} = W_0(r). \] (9)

**Solution method.** The problem is solved using the finite difference method. To implement the method, we introduce a division by spatial coordinate
\[ r_i = R_0 + (i-1)\Delta, \quad \Delta = \frac{h}{m} \quad (i = 1, \ldots, m + 1). \]  

(10)

We write equation (5), (6) in differential form at the internal points of division (10) \((i = 2, \ldots, m)\):

\[
\rho \frac{\partial^2 u}{\partial t^2} = \left( \frac{c_{33}}{\Delta^2} + \frac{1}{2\Delta} \left( c_{33}' + N \frac{c_{33}}{r_i} \right) \right) u_{i+1} + \left( N \frac{c_{13}}{r_i} - N \frac{c_{13}}{r_i^2} + N^2 \frac{c_{13} - a}{r_i^2} - 2 \frac{c_{33}}{\Delta^2} \right) u_i +
\]

\[
+ \left( \frac{c_{33}}{\Delta^2} - \frac{1}{2\Delta} \left( c_{33}' + N \frac{c_{33}}{r_i} \right) \right) u_{i-1} + \left( \frac{e_{33}'}{\Delta^2} + \frac{1}{2\Delta} \left( e_{33}' + N \frac{e_{33} - e_{31}}{r_i} \right) \right) \varphi_{i+1} - 2 \frac{e_{33}}{\Delta^2} \varphi_i + \left( \frac{e_{33}'}{\Delta^2} + \frac{1}{2\Delta} \left( e_{33}' + N \frac{e_{33} - e_{31}}{r_i} \right) \right) \varphi_{i-1},
\]

\[
= \left( e_{33} + \frac{\Delta}{2} \left( e_{33}' + N \frac{e_{33} + e_{33}}{r_i} \right) \right) u_{i+1} + \Delta^2 \left( N \frac{e_{31}'}{r_i} + N (N - 1) \frac{e_{31}}{r_i^2} - 2 \frac{e_{33}}{r_i} \right) u_i +
\]

\[
+ \left( e_{33} - \frac{\Delta}{2} \left( e_{33}' + N \frac{e_{31} + e_{33}}{r_i} \right) \right) u_{i-1}.
\]

It is necessary to exclude contour values of displacements from this system using one-sided difference expressions of the second order of accuracy:

\[
\left. \frac{du}{dr} \right|_{i=0} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta}, \quad \left. \frac{du}{dr} \right|_{i=m} = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta}.
\]

We also write conditions (7), (8) in difference form and find the values of unknown quantities at contour points

\[ u_1 = 0, \]

\[ u_{m+1} = -R_0 \frac{c_{33}}{3} (R_0) (-4u_m + u_{m-1} + e_{33}(R_0)(3\varphi_{m+1} - 4\varphi_m + \varphi_{m-1}) - 2\Delta \varphi_i), \]

\[ \varphi_1 = 0, \quad \varphi_{m+1} = V(t). \]  

(11)

After simple mathematical operations, a system of equations can be obtained and written in matrix form as follows

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{A} \mathbf{u} + \mathbf{B} \varphi, \]

\[ \mathbf{M} \varphi = \mathbf{C} \mathbf{u}, \]  

(12)

where \( \mathbf{u}, \varphi \) are the vectors of displacements and electrical potential in the internal points \( i = 2, \ldots, m \) of partitioning (10).

**Integration by time. Explicit numerical scheme.** We will look for the solution of the problem at times \( t^k \) with a timestep \( \Delta_t \). From the initial
conditions (9) we obtain a discretized distribution of displacements at zero ($t^0 = 0$) and first points of time ($t^1 = \Delta_t$):

$$u_i^0 = U_0(r_i), \quad u_i^1 = \Delta_t W_0(r_i) + U_0(r_i).$$

The distribution of the electrical potential at $t = t^1$ is determined from the second equation of the system (12):

$$\varphi^1 = M^{-1} C u^1.$$

At the following points of time ($t^k = k\Delta_t, \ k = 2, 3, \ldots$) the displacement at the internal points is found through displacement and electrical potential at previous moments of time:

$$u^{k+1} = 2u^k - u^{k-1} + \frac{\Delta_t^2}{\rho} \left( A u^k + B \varphi^k \right),$$

$$D \varphi^{k+1} = C u^{k+1}. \quad (13)$$

For convergence of the implicit scheme, it is necessary for the timestep be much smaller than a step in spatial coordinate. Therefore, we accept $\Delta_t = 0, 1\Delta_r$.

**Implicit scheme.** We use one-parameter Newmark scheme with parameter $\xi$ ($0.5 < \xi < 1$):

$$\dot{u}_i^{k+1} = \frac{u_i^{k+1} - u_i^k}{\xi \Delta_t} - \frac{1 - \xi}{\xi} \dot{u}_i^k, \quad \ddot{u}_i^{k+1} = \frac{u_i^{k+1} - u_i^k}{\xi^2 \Delta_t^2} - \frac{1}{\xi^2 \Delta_t} \ddot{u}_i^k - \frac{1 - \xi}{\xi} \dot{u}_i^k.$$

Equations (12) are written as

$$\frac{\rho}{\xi^2 \Delta_t^2} u^{k+1} - \left( A u^{k+1} + B \varphi^{k+1} \right) = \frac{\rho}{\xi^2 \Delta_t} \dot{u}^k + \rho \frac{1 - \xi}{\xi} \ddot{u}_i^k + \frac{\rho}{\xi^2 \Delta_t} u^k,$$

$$M u^{k+1} - D \varphi^{k+1} = 0. \quad (14)$$

We introduce a vector $y = (u, \varphi)$ and, taking into account (14), form a system of algebraic equations like $Cy = F$. For calculations start we need displacements, speed, potential and acceleration at zero point of time:

$$u_i^0 = U_0(r_i), \quad v_i^0 = W_0(r_i), \quad \varphi^0 = M^{-1} C u^0, \quad w^0 = \left( A u^0 + B \varphi^0 \right)/\rho.$$

The implicit scheme is absolutely stable, allowing us to take step by time and by the spatial coordinate of the same order.

**Numerical results.** We consider the piezoceramic flat layer, thick-walled cylinder and ball of piezoceramic PZT-4, the averaged material characteristics values of which are given in [8]. It is assumed that there is no mechanical load ($P_i = 0$), and to the conductive coatings of the piezoelements a stepped single electric signal ($V(t) = H(t)$, $H(t)$ is the Heaviside function) is applied. The geometry of piezoelements was determined by dimensionless parameters $R_0 = 1$ and $R_i = 2$ ($h = R_i - R_0 = 1$).
When modeling the functional heterogeneity of material characteristics (4), it was assumed that their distribution by thickness occurs according to a parabolic law with a multiplier

$$f(r) = ar^2 + br + c,$$  \hspace{1cm} (15)

relative to the table value.

The coefficients $a$, $b$ and $c$ of the law (15) were determined on the basis of the assumption that the average value of the function (4) is equal to one

$$\frac{1}{R_1 - R_0} \int_{R_0}^{R_1} f(r) dr = 1,$$  \hspace{1cm} (16)

on the middle surface of the layer the material characteristics are smaller by a parameter $y$ relative to the average value

$$f\left(\frac{R_0 + R_1}{2}\right) = 1 - y,$$  \hspace{1cm} (17)

and on the outer surfaces, they are inversely proportional to the area of electric rods:

$$R_1^N f(R_1) = R_0^N f(R_0),$$

what corresponds to the technological features of the piezoceramic manufacturing and polarization process.

Normalized relative to the tabular (averaged) material characteristics values, the thickness distribution laws (15) for flat ($N = 0$), cylindrical ($N = 1$) and spherical ($N = 2$) bodies at $y = 10\%$ are shown in Fig. 1.

Fig. 2(a) presents the displacement curves of the moving surface of the flat layer ($N = 0$), got from the system of equations (13) or (14) at different values of parameter $y$ (17). Dashed-dashed lines in this figure represent displacements at 20\%, obtained by explicit (13) and implicit (14) methods. In the implicit scheme the parameter $\xi$ was assumed equal to 0.6. From the comparison of these lines, it follows that the difference between them does not exceed 1\%. From the analysis of fig. 2(a), it can be seen that at the initial stage of oscillations, the effect of heterogeneity is not very noticeable, and with increasing time and at larger $y$ there is a slight change in the amplitude of oscillations and a decrease in the speed of wave propagation (for 20\%, this difference is approximately 2\%).
Fig. 2(b) illustrates the displacement of the moving surface of a hollow cylinder ($N = 1$) with different parameters of the parabolic distribution (15) for all material characteristics. The parabolic distribution considered in this paper leads to a decrease in the speed of propagation and a slight increase in the amplitude of oscillations with an increase in the parameter $y$. It should be noted that a cylinder with a rather large curvature is considered here, since for cylinders with a smaller curvature ($R_1/R_0 < 2$) the influence of the inhomogeneity of the material characteristics along the thickness becomes less noticeable.

Similar results and conclusions were obtained for the spherical layer (Fig. 2(c)). However, in this case, the increase in the amplitude of oscillations reaches 3% at $y = 20\%$.

It can be noted that with the considered version of the electromechanical load, oscillations occur in the compressed zone without entering the undeformed state ($u(h,t) < 0$).
An additional numerical analysis was also conducted to assess the influence of the law of the distribution of material characteristics by thickness $f(r)$ on the dynamic behavior of piezoelements. It was found that, under the condition of equality (16), this law has little effect on the graphs of the function $u(h,t)$.

However, due to the reduction of the scope of this work, these numerical results are not presented here.

**Conclusions.** The analysis of the numerical results shows that the heterogeneity of the material characteristics distribution by thickness can be neglected relative to their average values, since the deviation between the obtained results lies within acceptable limits for engineering calculations. Also, an important result is the confirmation of the assumption that for curved bodies such as hollow cylinders and spheres, the material characteristics can be considered constant in thickness, regardless of the curvature of the body.

The proposed technique can be applied for simulating of vibrations of different geometries bodies with significantly heterogeneous functional material or combined from several materials with a gradient transition between them.

**REFERENCES**

INFLUENCE OF MATERIAL FUNCTIONAL HETEROGENEITY ON NON-STATIONAR OSCILLATIONS OF PIEZOCERAMIC BODIES

The influence of material functional heterogeneity on mechanical oscillations of piezoelement under non-stationary electrical loading is investigated. Within the assumption of functional distribution of material characteristics by thickness of the piezoelectric element, which corresponds to the physical properties of the body, a unified system of solving equations was obtained to describe the thickness fluctuations of piezoelectric plates, cylinders, and balls. For controlling accuracy, the calculation is carried out using an explicit and implicit difference scheme.

Unsteady oscillations of a flat piezoceramic layer, cylinder, and sphere are investigated with a parabolic distribution of all material characteristics along the thickness of the element. It is assumed that the average value of the function along the thickness is equal to the tabular value of the material characteristic, and the value on electrodes is proportional to the area of electrodes. At such conditions, we obtained a decrease in the speed of disturbances propagation and a slight change in the amplitude associated with the curvature of the element. The increase in amplitude reaches 3% for balls. It should be noted that at given load oscillations occur in the compressed zone without entering the undeformed state. The considered cylinder and ball have a rather large curvature, for bodies with a smaller curvature the influence of the described effect will be smaller. The additional analysis indicates that the shape of the distribution curve under described above conditions also has little effect on the results.

It was established that the effect of functional heterogeneity within the same material has little effect on the oscillations of the piezoelement, that is, it is really possible to average the material characteristics by thickness at calculating, since the deviation between the results is within acceptable limits (up to 2.5%). Also, an important result is the confirmation of the assumption that for curved bodies such as cylinder and sphere, the material characteristics can be considered constant on thickness, regardless of the curvature of the body.

The proposed technique can be applied for studying of the vibrations of different geometries bodies with significantly heterogeneous functional material or what are combined from several materials with a gradient transition between them.

Key words: piezoceramics, electroelasticity, functional-gradient material, non-stationary oscillations.
Досліджуються коливання п'єзокерамічних плоских, циліндричних та сферичних тіл з урахуванням функціональної неоднорідності матеріалу.
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The oscillations of piezoceramic flat, cylindrical and spherical bodies are investigated taking into account the functional heterogeneity of material.
Fig. 2. Ref. 10.

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